

On the Decidability of Connectedness Constraints in 2D and 3D Euclidean Spaces

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IJCAI 2011 Barcelona

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Topological constraint languages

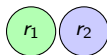
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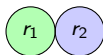
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The language $\mathcal{RCC8}$ [Randall et al.'92], [Egenhofer and Franzosa'91]

$DC(r_1, r_2)$



$EC(r_1, r_2)$



$PO(r_1, r_2)$



$EQ(r_1, r_2)$



$TPP(r_1, r_2)$



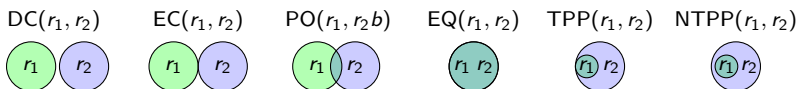
$NTPP(r_1, r_2)$



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Example of an $\mathcal{RCC8}$ -formula

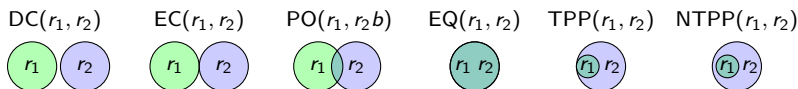
$$TPP(r_1, r_2) \wedge EC(r_2, r_3) \rightarrow (DC(r_1, r_3) \vee EC(r_1, r_3))$$



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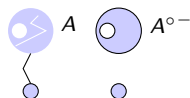


$\mathcal{RCC8}$ is insensitive to Euclidean spaces and is NP-complete.

[Renz'98], [Renz and Nebel'97]

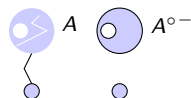
Regions in Euclidean spaces

The **regions** in $\mathbb{R}^n$ are the regular-closed sets in $\mathbb{R}^n$.
 $\text{RC}(\mathbb{R}^n)$ denotes the collection of regions in $\mathbb{R}^n$.
 $\text{RC}(\mathbb{R}^n)$ forms a Boolean algebra.

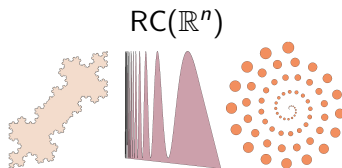


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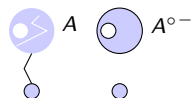


$\text{RCP}(\mathbb{R}^n)$ (polyhedra)

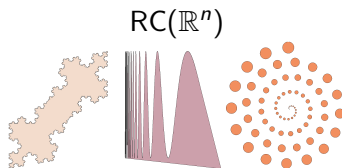


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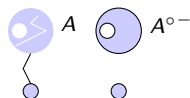
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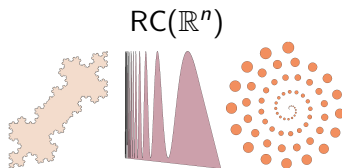
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For a topological language $\mathcal{L}$ and a collection of regions $\mathcal{M}$,
 $\text{Sat}(\mathcal{L}, \mathcal{M})$ denotes the $\mathcal{L}$ -formulas satisfiable by regions in $\mathcal{M}$.

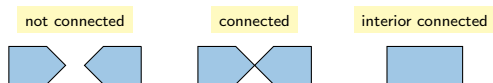
Languages with connectedness

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Connectedness and Interior-Connectedness

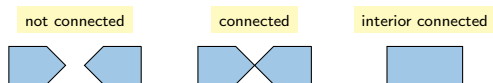
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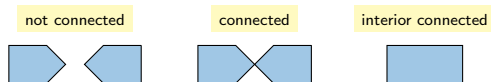
Connectedness and Interior-Connectedness



Languages [Kontchakov et al. 2008]

Languages with connectedness

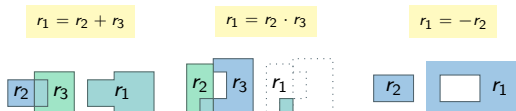
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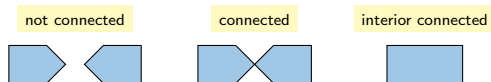
$$\mathcal{Bc} = (c, +, \cdot, -, =)$$

$$\mathcal{Bc}^{\circ} = (c^{\circ}, +, \cdot, -, =)$$



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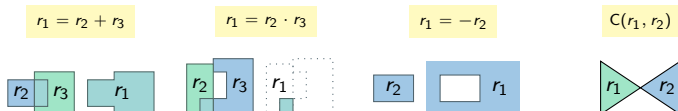
Languages [Kontchakov et al. 2008]

$$\mathcal{B}c = (c, +, \cdot, -, =)$$

$$\mathcal{B}c^\circ = (c^\circ, +, \cdot, -, =)$$

$$\mathcal{C}c = (C, c, +, \cdot, -, =)$$

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Regions with **finitely** many components:

Regions with infinitely many components

Regions with *finitely* many components:

(1) $c(r)$



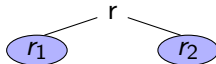
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Regions with **finitely** many components:

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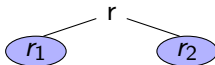
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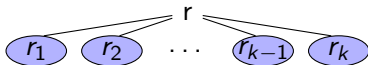
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(k) $r = \sum_{i=1}^k r_i \wedge \bigwedge_{i=1}^k (c(r_i) \wedge r_i > 0) \wedge \bigwedge_{1 \leq i < j \leq k} \neg C(r_i, r_j)$



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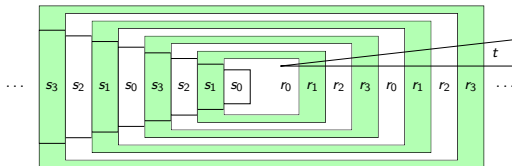
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Regions with **infinitely** many components:

There exists a $\mathcal{C}c$ -formula ψ_∞ with variables $r_0, \dots, r_3, s_0, \dots, s_3, t$ that is satisfiable in $\text{RC}(\mathbb{R}^n)$ but only by tuples featuring regions with infinitely many components.



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Theorem

If $\mathcal{L} \in \{\mathcal{C}c, \mathcal{C}c^\circ, \mathcal{B}c\}$, then

$$\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n)) \neq \text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^n))$$

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Encoding the Post Correspondence Problem

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A PCP instance P

100	00	11
0	01	110
1	2	3

A solution to P

11	00	11	100
110	01	110	0
3	2	3	1

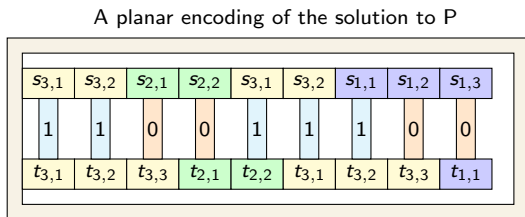
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- ▶ Every solution to P can be encoded by polygons



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Theorem

If $\mathcal{L} \in \{\mathcal{C}c, \mathcal{C}c^\circ, \mathcal{B}c, \mathcal{B}c^\circ\}$, then the problems $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^2))$ and $\text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^2))$ are r.e.-hard.

The third dimension

$$\mathcal{B}c^\circ$$

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The problem $Sat(\mathcal{B}c^\circ, RCP(\mathbb{R}^3))$ is EXPTIME-complete.

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[Kontchakov et al. 2008]

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Open Problems

Are $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^3))$ and $\text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^3))$ decidable?

Conclusions

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- ▶ Sensitive to Euclidean spaces of different dimensions.
- ▶ Sensitive to tameness:
 - ▶ regions with **infinitely many components** ($\mathcal{C}_c$, $\mathcal{C}_c^\circ$ and $\mathcal{B}_c$);
 - ▶ regions with **pathological boundaries** ($\mathcal{B}_c^\circ$).
- ▶ Satisfiability of formulas over regions in $\mathbb{R}^2$ is **undecidable**.
- ▶ Satisfiability of $\mathcal{B}_c^\circ$ -formulas over regions in $\mathbb{R}^3$ is decidable.
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Open Problems

- ▶ Determine if the satisfiability of $\{\mathcal{C}c, \mathcal{C}c^\circ, \mathcal{B}c\}$ -formulas in $\mathbb{R}^3$ is **decidable**?
- ▶ Identify **decidable** fragments of the languages with connectedness.