

Cooperative Games in Multiagent Systems

AAMAS'11

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Tutorial Overview

- Introduction
- Definitions
- Solution concepts
- Representations and computational issues
- Coalition formation in multiagent systems research
- Bayesian models of coalitional games
- Other topics of interest

Game Theory in MAS

- Multi-agent systems research area studies interactions between self-interested computational entities
- Game theory: a branch of economics that deals with decision-making in environments full of self-interested entities
- Game theory proposes solution concepts, defining rational outcomes
- However, solution concepts may be hard to compute...

A Point of Reference: Non-Cooperative Games

- You are probably all familiar with non-cooperative games...
- A non-cooperative game is defined by
 - a set of agents (players) $N = \{1, \dots, n\}$
 - for each agent $i \in N$, a set of actions S_i
 - for each agent $i \in N$, a utility function u_i ,
 $u_i: S_1 \times \dots \times S_n \rightarrow \mathbb{R}$
- Observe that an agent's utility depends not just on her action, but on actions of other agents
 - thus, for i finding the best action involves deliberating about what other will do

Example: Prisoner's Dilemma



- Two agents committed a crime.
- Court does not have enough evidence to convict them of the crime, but can convict them of a minor offence (1 year in prison each)
- If one suspect confesses (acts as an informer), he walks free, and the other suspect gets 4 years
- If both confess, each gets 3 years
- Agents have no way of communicating or making binding agreements

Prisoner's Dilemma: Matrix Representation

		P2	
		quiet	confess
P1	quiet	$(-1, -1)$	$(-4, 0)$
	confess	$(0, -4)$	$(-3, -3)$

- Interpretation: the pair (x, y) at the intersection of row i and column j means that the row player gets x and the column player gets y

Prisoners' Dilemma: the Rational Outcome

- P1's reasoning:
 - if P2 stays quiet, I should confess
 - if P2 confesses, I should confess, too
- P2 reasons in the same way
- Result: both confess and get 3 years in prison.
- However, if they chose to cooperate and stay quiet, they could get away with 1 year each.
- So why do not they cooperate?

		P2	
		Q	C
P1	Q	$(-1, -1)$	$(-4, 0)$
	C	$(0, -4)$	$(-3, -3)$

Assumptions in Non-Cooperative Games

- Cooperation does not occur in prisoners' dilemma, because players cannot make binding agreements
- But what if binding agreements are possible?
- This is exactly the class of scenarios studied by cooperative game theory

Cooperative Games

- Cooperative games model scenarios, where
 - agents can benefit by cooperating
 - binding agreements are possible
- In cooperative games, actions are taken by groups of agents

Transferable utility games:
 payoffs are given to the group and then divided among its members

Non-transferable utility games: group actions result in payoffs to individual group members

Phases of Coalitional Action

- Agents form coalitions (teams)
- Each coalition chooses its action
- Non-transferable utility (NTU) games: the choice of coalitional actions (by all coalitions) determines each player's payoffs
- Transferable utility (TU) games: the choice of coalitional actions (by all coalitions) determines the payoff of each coalition
 - the members of the coalition then need to divide this joint payoff

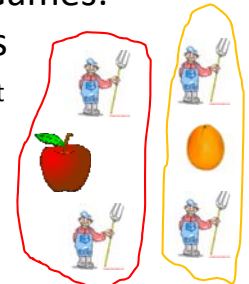
Non-Transferable Utility Games: Writing Papers

- n researchers working at n different universities can form groups to write papers on game theory
- each group of researchers can work together; the composition of a group determines the quality of the paper they produce
- each author receives a payoff from his own university
 - promotion
 - bonus
 - teaching load reduction
- Payoffs are non-transferable



Transferable Utility Games: Happy Farmers

- n farmers can cooperate to grow fruit
- Each group of farmers can grow apples or oranges
- a group of size k can grow $f(k)$ tons of apples and $g(k)$ tons of oranges
 - $f()$, $g()$ are convex functions of k
- Fruit can be sold in the market:
 - if there are x tons of apples and y tons of oranges on the market, the market prices for apples and oranges are $\max\{X - x, 0\}$ and $\max\{Y - y, 0\}$, respectively
 - X, Y are some large enough constants
- The profit of each group depends on the quantity and type of fruit it grows, and the market price



Transferable Utility Games: Buying Ice-Cream

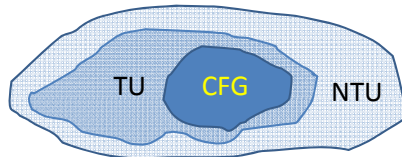
- n children, each has some amount of money
 - the i -th child has b_i dollars
 - three types of ice-cream tubs are for sale:
 - Type 1 costs \$7, contains 500g
 - Type 2 costs \$9, contains 750g
 - Type 3 costs \$11, contains 1kg
- 
- children have utility for ice-cream, and do not care about money
 - The payoff of each group: the maximum quantity of ice-cream the members of the group can buy by pooling their money
 - The ice-cream can be shared arbitrarily within the group

Characteristic Function Games vs. Partition Function Games

- In general TU games, the payoff obtained by a coalition depends on the actions chosen by other coalitions
 - these games are also known as partition function games (PFG)
- Characteristic function games (CFG): the payoff of each coalition only depends on the action of that coalition
 - in such games, each coalition can be identified with the profit it obtains by choosing its best action
 - Ice Cream game is a CFG
 - Happy Farmers game is a PFG, but not a CFG

Classes of Cooperative Games: The Big Picture

- Any TU game can be represented as an NTU game with a continuum of actions
 - each payoff division scheme in the TU game can be interpreted as an action in the NTU game



- We will focus on characteristic function games, and use term “TU games” to refer to such games

How Is a Cooperative Game Played?

- Even though agents work together they are still selfish
- The partition into coalitions and payoff distribution should be such that no player (or group of players) has an incentive to deviate
- We may also want to ensure that the outcome is fair: the payoff of each agent is proportional to his contribution
- We will now see how to formalize these ideas

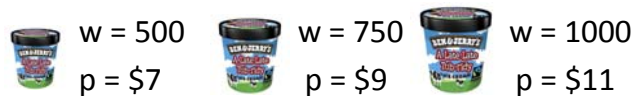
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Transferable Utility Games Formalized

- A **transferable utility game** is a pair (N, v) , where:
 - $N = \{1, \dots, n\}$ is the set of **players**
 - $v: 2^N \rightarrow \mathbb{R}$ is the **characteristic function**
 - for each subset of players C , $v(C)$ is the amount that the members of C can earn by working together
 - usually it is assumed that v is
 - normalized: $v(\emptyset) = 0$
 - non-negative: $v(C) \geq 0$ for any $C \subseteq N$
 - monotone: $v(C) \leq v(D)$ for any C, D such that $C \subseteq D$
- A **coalition** is any subset of N ;
 N itself is called the **grand coalition**

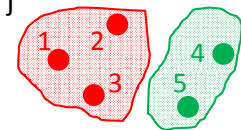
Ice-Cream Game: Characteristic Function



- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 750, v(\{C, P\}) = 750, v(\{M, P\}) = 500$
- $v(\{C, M, P\}) = 1000$

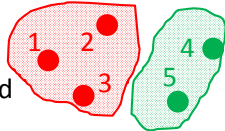
Transferable Utility Games: Outcome

- An **outcome** of a TU game $G = (N, v)$ is a pair $(CS, \underline{x})$, where:
 - $CS = (C_1, \dots, C_k)$ is a **coalition structure**,
 i.e., **partition** of N into coalitions:
 - $\cup_i C_i = N, C_i \cap C_j = \emptyset$ for $i \neq j$
 - $\underline{x} = (x_1, \dots, x_n)$ is a **payoff vector**,
 which distributes the value of each coalition in CS :
 - $x_i \geq 0$ for all $i \in N$
 - $\sum_{i \in C} x_i = v(C)$ for each C in CS



Transferable Utility Games: Outcome

- Example:
 - suppose $v(\{1, 2, 3\}) = 9$, $v(\{4, 5\}) = 4$
 - then $((\{1, 2, 3\}, \{4, 5\}), (3, 3, 3, 3, 1))$ is an outcome
 - $((\{1, 2, 3\}, \{4, 5\}), (2, 3, 2, 3, 3))$ is NOT an outcome: transfers between coalitions are not allowed
- An outcome $(CS, \underline{x})$ is called an **imputation** if it satisfies **individual rationality**: $x_i \geq v(\{i\})$ for all $i \in N$
- Notation: we will denote $\sum_{i \in C} x_i$ by $x(C)$



Superadditive Games

- **Definition**: a game $G = (N, v)$ is called superadditive if $v(C \cup D) \geq v(C) + v(D)$ for any two disjoint coalitions C and D
- Example: $v(C) = |C|^2$:
 - $v(C \cup D) = (|C| + |D|)^2 \geq |C|^2 + |D|^2 = v(C) + v(D)$
- In superadditive games, two coalitions can always **merge** without losing money; hence, we can assume that players form the **grand coalition**

Superadditive Games

- **Convention**: in superadditive games, we identify outcomes with payoff vectors for the grand coalition
 - i.e., an outcome is a vector $\underline{x} = (x_1, \dots, x_n)$ with $\sum_{i \in N} x_i = v(N)$
- **Caution**: some GT/MAS papers define outcomes in this way even if the game is not superadditive
- Any non-superadditive game $G = (N, v)$ can be transformed into a superadditive game $G^{SA} = (N, v^{SA})$ by setting $v^{SA}(C) = \max_{(C_1, \dots, C_k) \in P(C)} \sum_{i=1, \dots, k} v(C_i)$, where $P(C)$ is the space of all partitions of C
- G^{SA} is called the **superadditive cover** of G

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- **Solution concepts**
 - core
 - least core
 - cost of stability
 - nucleolus
 - bargaining set
 - kernel
 - Shapley value and Banzhaf index
- Representations and computational issues
- Coalition formation in MAS research
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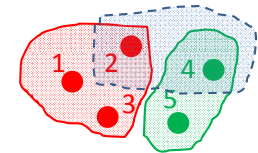
What Is a Good Outcome?

-  C: \$4,  M: \$3,  P: \$3
- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 500, v(\{C, P\}) = 500, v(\{M, P\}) = 0$
- $v(\{C, M, P\}) = 750$
- This is a superadditive game
 - outcomes are payoff vectors
- How should the players share the ice-cream?
 - if they share as (200, 200, 350), Charlie and Marcie can **get more** ice-cream by buying a 500g tub on their own, and **splitting** it equally
 - the outcome (200, 200, 350) is not **stable**!

Transferable Utility Games: Stability

- Definition:** the **core** of a game is the set of all **stable** outcomes, i.e., outcomes that no coalition wants to deviate from

$$\text{core}(G) = \{(CS, \underline{x}) \mid \sum_{i \in C} x_i \geq v(C) \text{ for any } C \subseteq N\}$$
 - each coalition earns at least as much as it can make on its own
- Note that G is **not** assumed to be superadditive
- Example
 - suppose $v(\{1, 2, 3\}) = 9$, $v(\{4, 5\}) = 4$, $v(\{2, 4\}) = 7$
 - then $((\{1, 2, 3\}, \{4, 5\}), (3, 3, 3, 3, 1))$ is NOT in the core



Ice-Cream Game: Core

-  C: \$4,  M: \$3,  P: \$3
- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0, v(\{C, M, P\}) = 750$
- $v(\{C, M\}) = 500, v(\{C, P\}) = 500, v(\{M, P\}) = 0$
- (200, 200, 350) is not in the core:
 - $v(\{C, M\}) > x_C + x_M$
- (250, 250, 250) is in the core:
 - no subgroup of players can deviate so that each member of the subgroup gets more
- (750, 0, 0) is also in the core:
 - Marcie and Pattie cannot get more on their own!

Games with Empty Core

- The core is a very attractive solution concept
- However, some games have empty cores
- $G = (N, v)$
 - $N = \{1, 2, 3\}, v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise
 - consider an outcome $(CS, \underline{x})$
 - if $CS = (\{1\}, \{2\}, \{3\})$, the grand coalition can deviate
 - if $CS = (\{1, 2\}, \{3\})$, either 1 or 2 gets less than 1, so can deviate with 3
 - same argument for $CS = (\{1, 3\}, \{2\})$ or $CS = (\{2, 3\}, \{1\})$
 - suppose $CS = \{1, 2, 3\}$:
 - $x_i > 0$ for some i , so $x(N \setminus \{i\}) < 1$, yet $v(N \setminus \{i\}) = 1$

Core and Superadditivity

- Suppose the game is **not superadditive**, but the outcomes are defined as payoff vectors for the **grand coalition**
- Then the core may be **empty**, even if according to the **standard** definition it is not
- $G = (N, v)$
 - $N = \{1, 2, 3, 4\}$, $v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise
 - not superadditive: $v(\{1, 2\}) + v(\{3, 4\}) = 2 > v(\{1, 2, 3, 4\})$
 - no payoff vector for the grand coalition is in the core: either $\{1, 2\}$ or $\{3, 4\}$ get less than 1, so can deviate
 - $((\{1, 2\}, \{3, 4\}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}))$ is in the core

ϵ -Core

- If the core is empty, we may want to find **approximately stable** outcomes
- Need to **relax** the notion of the core:
 - core: $(CS, \underline{x}): x(C) \geq v(C)$ for all $C \subseteq N$
 - ϵ -core: $(CS, \underline{x}): x(C) \geq v(C) - \epsilon$ for all $C \subseteq N$
- Is usually defined for superadditive games only
- Example: $G = (N, v)$, $N = \{1, 2, 3\}$,
 - $v(C) = 1$ if $|C| > 1$, $v(C) = 0$ otherwise
 - $1/3$ -core is non-empty: $(1/3, 1/3, 1/3) \in 1/3$ -core
 - ϵ -core is empty for any $\epsilon < 1/3$:
 - $x_i \geq 1/3$ for some $i = 1, 2, 3$, so $x(N \setminus \{i\}) \leq 2/3$, $v(N \setminus \{i\}) = 1$

Least Core

- If an outcome $(CS, \underline{x})$ is in ϵ -core, the **deficit** $v(C) - x(C)$ of any coalition is at most ϵ
- We are interested in outcomes that **minimize** the **worst-case** deficit
- Let $\epsilon^*(G) = \inf \{ \epsilon \mid \epsilon\text{-core of } G \text{ is not empty} \}$
 - it can be shown that $\epsilon^*(G)$ -core is not empty
- Definition:** $\epsilon^*(G)$ -core is called the **least core** of G
 - $\epsilon^*(G)$ is called the **value** of the least core
- $G = (N, v)$, $N = \{1, 2, 3\}$,
 - $v(C) = 1$ if $|C| > 1$, $v(C) = 0$ otherwise
 - $1/3$ -core is non-empty, but ϵ -core is empty for any $\epsilon < 1/3$, so least core = $1/3$ -core

Cost of Stability [Bachrach et al.'09]

- If the core of a superadditive game is empty, an external party may want to **stabilize** the game by offering **subsidies** to the grand coalition
 - in non-superadditive games, one may want to subsidize a particular coalition structure
- Given a game $G = (N, v)$, let $G^\Delta = (N, v^\Delta)$, where $v^\Delta(N) = v(N) + \Delta$ and $v^\Delta(C) = v(C)$ for any $C \neq N$
- Definition:** the **Cost of Stability (CoS)** of a superadditive game G is
 - $CoS(G) = \inf \{ \Delta \geq 0 \mid G^\Delta \text{ has a non-empty core} \}$
- $CoS(G) > 0$ iff $\epsilon^*(G) > 0$
 - however, $CoS(G)$ and $\epsilon^*(G)$ may differ by a factor of n

Advanced Solution Concepts

- We will now define 5 more solution concepts:
 - the nucleolus
 - the bargaining set
 - the kernel
 - the Shapley value
 - the Banzhaf index
- more complicated stability considerations
- fairness considerations
- For simplicity, we will define all these solution concepts for superadditive games only
 - however, all definitions generalize to non-superadditive games

Nucleolus

- Least core is always non-empty
- However, it may contain more than one point
- Can we identify the **most stable** point in the least core?
- Given an outcome $\underline{x}$ of a game $G(N, v)$, let $d(C) = v(C) - x(C)$
 - $d(C)$ is called the deficit of C wrt $\underline{x}$
- The least core minimizes the **max** deficit
- The nucleolus of G is an imputation that
 - minimizes the **max** deficit
 - given this, minimizes the **2nd-largest** deficit, etc.

Nucleolus: Formal Definition and Properties

- Definition: the **deficit vector** for an outcome $\underline{x}$ is the list of deficits of all 2^n coalitions, ordered from the **largest** to the **smallest**
- Definition: the **nucleolus** is an **imputation** that corresponds to the **lexicographically smallest** deficit vector
- If we optimize over **all outcomes** (and not just imputations), we obtain **pre-nucleolus**
- Nucleolus is **unique**
 - the “most stable” outcome
- Appears in Talmud as an estate division scheme

Objections and Counterobjections

- An outcome is not in the core if some coalition objects to it; but is the objection itself **plausible**?
- Fix an imputation $\underline{x}$ for a game $G=(N, v)$
- A pair $(\underline{y}, S)$, where $\underline{y}$ is an imputation and $S \subseteq N$, is an **objection** of player i against player j to $\underline{x}$ if
 - $i \in S, j \notin S, y(S) = v(S)$
 - $y_k > x_k$ for all $k \in S$
- A pair $(\underline{z}, T)$, where $\underline{z}$ is an imputation and $T \subseteq N$, is a **counterobjection** to the objection $(\underline{y}, S)$ if
 - $j \in T, i \notin T, z(S) = v(S), T \cap S \neq \emptyset$
 - $z_k \geq x_k$ for all $k \in T \setminus S$
 - $z_k \geq y_k$ for all $k \in T \cap S$

Bargaining Set

- An objection is said to be **justified** if it does not admit a counterobjection
- **Definition**: the **bargaining set** of a game G consist of all imputations that do not admit a **justified objection**
- The core is the set of all imputations that do not admit an **objection**. Hence
core $\subseteq$ bargaining set

Kernel (1/2)

- Let $I(i, k) = \{C \subseteq N \mid i \in C, k \notin C\}$
- **Definition**: the **surplus** $S(i, k)$ of an agent i over agent k wrt an imputation $\underline{x}$ is
 $S(i, k) = \max_{C \in I(i, k)} d(C)$, where $d(C) = v(C) - x(C)$
- **Definition**: an agent i **outweighs** agent k under $\underline{x}$ if $S(i, k) > S(k, i)$
- If i outweighs k under $\underline{x}$,
 i should be able to claim some of k 's payoff x_k
- However, the amount he can claim is limited by **individual rationality**: we should have $x_k \geq v(\{k\})$

Kernel (2/2)

- **Definition**: two agents i and k are in **equilibrium** with respect to the imputation $\underline{x}$ if one of the following holds:
 - $S(i, k) = S(k, i)$
 - $S(i, k) > S(k, i)$ and $x_k = v(\{k\})$
 - $S(i, k) < S(k, i)$ and $x_i = v(\{i\})$
- **Definition**: an imputation $\underline{x}$ is in the **kernel** if any two agents i and k are in equilibrium wrt $\underline{x}$
nucleolus $\subseteq$ kernel $\subseteq$ bargaining set

Stability vs. Fairness

- Outcomes in the core may be **unfair**
- $G = (N, v)$
 - $N = \{1, 2\}$, $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
- $(15, 5)$ is in the core:
 - player 2 cannot benefit by deviating
- However, this is unfair since 1 and 2 are **symmetric**
- How do we divide payoffs in a fair way?

Marginal Contribution

- A fair payment scheme would reward each agent according to his **contribution**
- First attempt: given a game $G = (N, v)$, set $x_i = v(\{1, \dots, i-1, i\}) - v(\{1, \dots, i-1\})$
 - payoff to each player = his **marginal contribution to the coalition of his predecessors**
- We have $x_1 + \dots + x_n = v(N)$
 - x is a payoff vector
- However, payoff to each player depends on the order
- $G = (N, v)$
 - $N = \{1, 2\}$, $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
 - $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$

Average Marginal Contribution

- Idea:** to remove the dependence on ordering, can **average** over all possible orderings
- $G = (N, v)$
 - $N = \{1, 2\}$, $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
 - **1, 2:** $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$
 - **2, 1:** $y_2 = v(2) - v(\emptyset) = 5$, $y_1 = v(\{1, 2\}) - v(\{2\}) = 15$
 - $z_1 = (x_1 + y_1)/2 = 10$, $z_2 = (x_2 + y_2)/2 = 10$
 - the resulting outcome is fair!
- Can we generalize this idea?

Shapley Value

- Reminder: a **permutation** of $\{1, \dots, n\}$ is a one-to-one mapping from $\{1, \dots, n\}$ to itself
 - let $P(N)$ denote the set of all permutations of N
- Let $S_\pi(i)$ denote the set of predecessors of i in $\pi \in P(N)$



- For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$
- Definition:** the **Shapley value** of player i in a game $G = (N, v)$ with $|N| = n$ is

$$\phi_i(G) = 1/n! \sum_{\pi: \pi \in P(N)} \delta_i(S_\pi(i))$$

- In the previous slide we have $\phi_1 = \phi_2 = 10$

Shapley Value:

Probabilistic Interpretation

- ϕ_i is i 's **average marginal contribution** to the coalition of its predecessors, over all permutations
- Suppose that we choose a permutation of players uniformly at random, among all possible permutations of N
 - then ϕ_i is the **expected marginal contribution** of player i to the coalition of his predecessors

Shapley Value: Properties (1)

- Proposition: in any game G , $\phi_1 + \dots + \phi_n = v(N)$
 – $(\phi_1, \dots, \phi_n)$ is a payoff vector
- Proof:
 Given a permutation π , let π_i denote $\pi^{-1}(i)$. Then

$$\sum_{i=1, \dots, n} \phi_i =$$

$$\frac{1}{n!} \sum_{\pi: \pi \in P(N)} \sum_{i=1, \dots, n} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] =$$

$$\frac{1}{n!} \sum_{\pi: \pi \in P(N)} \sum_{i=1, \dots, n} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] =$$

$$\frac{1}{n!} \sum_{\pi: \pi \in P(N)} [v(\{\pi_1\}) - v(\emptyset) + \dots + v(N) - v(N \setminus \{\pi_n\})] =$$

$$\frac{1}{n!} \sum_{\pi: \pi \in P(N)} v(N) =$$

$$v(N)$$

$\pi_1 \pi_2 \dots \pi_n$

Shapley Value: Properties (2)

- Definition: a player i is a **dummy** in a game $G = (N, v)$ if $v(C) = v(C \cup \{i\})$ for any $C \subseteq N$
- Proposition: if a player i is a dummy in a game $G = (N, v)$ then $\phi_i = 0$
- Proof: if i is a dummy, all summands in $\sum_{\pi: \pi \in P(N)} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]$ equal **0**
 – converse is **only** true if the game is **monotone**:
 - $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 1$, $v(\emptyset) = v(\{1, 2\}) = 0$
 - $\phi_1 = \phi_2 = 0$, but **1** and **2** are not dummies

Shapley Value: Properties (3)

- Definition: given a game $G = (N, v)$, two players i and j are said to be **symmetric** if $v(C \cup \{i\}) = v(C \cup \{j\})$ for any $C \subseteq N \setminus \{i, j\}$
- Proposition: if i and j are symmetric then $\phi_i = \phi_j$
- Proof sketch:
 - given a permutation π , let π' denote the permutation obtained from π by swapping i and j
 - mapping $\pi \rightarrow \pi'$ is a one-to-one mapping
 - we can show $\delta_i(S_\pi(i)) = \delta_j(S_{\pi'}(j))$

$\dots i \dots j \dots \rightarrow \dots j \dots i \dots$

Shapley Value: Properties (4)

- Definition: Let $G_1 = (N, u)$ and $G_2 = (N, v)$ be two games with the same set of players. Then $G = G_1 + G_2$ is the game with the set of players N and characteristic function w given by $w(C) = u(C) + v(C)$ for all $C \subseteq N$
- Proposition: $\phi_i(G_1 + G_2) = \phi_i(G_1) + \phi_i(G_2)$
- Proof: $\phi_i(G_1 + G_2) =$

$$\frac{1}{n!} \sum_{\pi} [u(S_\pi(i) \cup \{i\}) + v(S_\pi(i) \cup \{i\}) - u(S_\pi(i)) - v(S_\pi(i))]$$

$$= \frac{1}{n!} \sum_{\pi} [u(S_\pi(i) \cup \{i\}) - u(S_\pi(i))] +$$

$$\frac{1}{n!} \sum_{\pi} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] = \phi_i(G_1) + \phi_i(G_2)$$

Axiomatic Characterization

- **Properties** of Shapley value:
 1. Efficiency: $\phi_1 + \dots + \phi_n = v(N)$
 2. Dummy: if i is a dummy, $\phi_i = 0$
 3. Symmetry: if i and j are symmetric, $\phi_i = \phi_j$
 4. Additivity: $\phi_i(G_1 + G_2) = \phi_i(G_1) + \phi_i(G_2)$
- **Theorem**: Shapley value is the **only** payoff distribution scheme that has properties (1) - (4)

Banzhaf Index

- Instead of averaging over all permutations of players, we can average over all coalitions
- **Definition**: the **Banzhaf index** of player i in a game $G = (N, v)$ with $|N| = n$ is

$$\beta_i(G) = 1/2^{n-1} \sum_{C \subseteq N \setminus \{i\}} \delta_i(C)$$
- Satisfies dummy axiom, symmetry and additivity
- However, may fail efficiency: it may happen that $\sum_{i \in N} \beta_i \neq v(N)$

Shapley and Banzhaf: Examples

- Example 1 (unanimity game):
 - $G = (N, v)$, $|N| = n$, $v(C) = 1$ if $C = N$, $v(C) = 0$ otherwise
 - $\delta_i(C) = 1$ iff $C = N \setminus \{i\}$
 - $\phi_i(G) = (n-1)!/n! = 1/n$ for $i = 1, \dots, n$
 - $\beta_i(G) = 1/2^{n-1}$ for $i = 1, \dots, n$
- Example 2 (majority game):
 - $G = (N, v)$, $|N| = 2k$, $v(C) = 1$ if $|C| > k$, $v(C) = 0$ otherwise
 - $\delta_i(C) = 1$ iff $|C| = k$
 - $\phi_i(G) = k!(n-k-1)!/n! = 1/n$ for $i = 1, \dots, n$
 - $\beta_i(G) = 1/2^{n-1} \times (2k)!/(k!)^2 \approx 2/\sqrt{\pi k}$ for $i = 1, \dots, n$

Tutorial Overview

- Introduction
- Definitions
- Solution concepts
- **Representations and computational issues**
- Coalition formation in multiagent systems research
- Bayesian models of coalitional games
- Other topics of interest

Computational Issues in Coalitional Games

- We have defined many solution concepts - but can we compute them **efficiently**?
- Problem: the **naive** representation of a coalitional game is **exponential** in the number of players **n**
 - need to **list values** of all coalitions
- We are usually interested in algorithms whose running time is **polynomial** in **n**
- So what can we do?

How to Deal with Representation Issues?

- Strategy 1: oracle representation
 - assume that we have a **black-box poly-time** algorithm that, given a coalition $C \subseteq N$, outputs its value $v(C)$
 - for some **special classes** of games, this allows us compute some solution concepts using **polynomially** many queries
- Strategy 2: restricted classes
 - consider games on **combinatorial structures**
 - **problem**: not all games can be represented in this way
- Strategy 3: give up on worst-case succinctness
 - devise **complete** representation languages that allow for **compact** representation of **interesting** games

Tutorial Overview

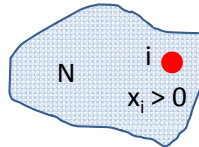
- Introduction
- Definitions
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 - **oracle representation**
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 - weighted voting games
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Simple Games

- Definition: a game $G = (N, v)$ is **simple** if
 - $v(C) \in \{0, 1\}$ for any $C \subseteq N$
 - v is **monotone**: if $v(C) = 1$ and $C \subseteq D$, then $v(D) = 1$
- A coalition C in a simple game is said to be **winning** if $v(C) = 1$ and **losing** if $v(C) = 0$
- Definition: in a simple game, a player i is a **veto** player if $v(C) = 0$ for any $C \subseteq N \setminus \{i\}$
 - equivalently, by monotonicity, $v(N \setminus \{i\}) = 0$
- Traditionally, in simple games an outcome is identified with a payoff vector for **N**
- Theorem: a simple game has a non-empty core iff it has a veto player.

Simple Games: Characterization of the Core

- Proof ($\Leftarrow$):
 - suppose i is a veto player
 - consider a payoff vector $\underline{x}$ with $x_i = 1$, $x_k = 0$ for $k \neq i$
 - no coalition C can deviate from $\underline{x}$:
 - if $i \in C$, we have $\sum_{k \in C} x_k = 1 \geq v(C)$
 - if $i \notin C$, we have $v(C) = 0$
- Proof ($\Rightarrow$):
 - consider an arbitrary payoff vector $\underline{x}$:
 - we have $\sum_{k \in N} x_k = v(N) = 1$; thus $x_i > 0$ for some $i \in N$
 - but then $N \setminus \{i\}$ can deviate:
 - since i is not a veto, $v(N \setminus \{i\}) = 1$, yet $x(N \setminus \{i\}) = 1 - x_i < 1$

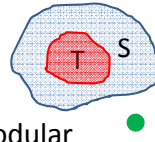
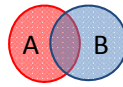


Simple Games: Checking Non-Emptiness of the Core

- Corollary: in a simple game G , a payoff vector $\underline{x}$ is in the core iff $x_i = 0$ for any non-veto player i
 - proved similarly
- Checking if a player i is a veto player is easy
 - a single oracle access to compute $v(N \setminus \{i\})$
- Thus, in simple games
 - checking non-emptiness of the core or
 - checking if a given outcome is in the core
 is easy given oracle access to the characteristic function
 - this is no longer the case if we allow coalition structures

Convex Games

- Definition: a function $f: 2^N \rightarrow \mathbb{R}$ is called **supermodular** if $f(\emptyset) = 0$ and $f(A \cup B) + f(A \cap B) \geq f(A) + f(B)$ for any $A, B \subseteq N$ (not necessarily disjoint)
 - any supermodular function is superadditive, but the converse is not true
- Proposition: if f is supermodular, $T \subset S$, and $i \notin S$, then $f(T \cup \{i\}) - f(T) \leq f(S \cup \{i\}) - f(S)$
 - a player is more useful when he joins a bigger coalition
- Definition: a game $G = (N, v)$ is **convex** if its characteristic function is supermodular



Convex Games: Non-Emptiness of The Core

- Proposition: any convex game has a **non-empty core**
- Proof:
 - set $x_1 = v(\{1\})$,
 $x_2 = v(\{1, 2\}) - v(\{1\})$,
 $\dots$
 $x_n = v(N) - v(N \setminus \{n\})$
 - i.e., pay each player his marginal contribution to the coalition formed by his predecessors
 - $\underline{x}$ is a payoff vector: $x_1 + x_2 + \dots + x_n = v(\{1\}) + v(\{1, 2\}) - v(\{1\}) + \dots + v(N) - v(N \setminus \{n\}) = v(N)$
 - remains to show that $(x_1, x_2, \dots, x_n)$ is in the core

Convex Games Have Non-Empty Core

- Proof (continued):
 - $x_1 = v(\{1\})$, $x_2 = v(\{1, 2\}) - v(\{1\})$, ..., $x_n = v(N) - v(N \setminus \{n\})$
 - pick any coalition $C = \{i, j, \dots, s\}$, where $i < j < \dots < s$
 - we will prove $v(C) \leq x_i + x_j + \dots + x_s$, i.e., C cannot deviate
 - $v(C) = v(\{i\}) + v(\{i, j\}) - v(\{i\}) + \dots + v(C) - v(C \setminus \{s\})$
 - $v(\{i\}) = v(\{i\}) - v(\emptyset) \leq v(\{1, \dots, i-1, i\}) - v(\{1, \dots, i-1\}) = x_i$
 - $v(\{i, j\}) - v(\{i\}) \leq v(\{1, \dots, j-1, j\}) - v(\{1, \dots, j-1\}) = x_j$
 -
 - $v(C) - v(C \setminus \{s\}) \leq v(\{1, \dots, s-1, s\}) - v(\{1, \dots, s-1\}) = x_s$
 - thus, $v(C) \leq x_i + x_j + \dots + x_s$



Convex Games: Remarks

- This proof suggests a simple algorithm for **constructing** an outcome in the core
 - order the players as **1, ..., n**
 - query the oracle for $v(\{1\})$, $v(\{1, 2\})$, ..., $v(N)$
 - set $x_i = v(\{1, \dots, i-1, i\}) - v(\{1, \dots, i-1\})$
- This argument also shows that for convex games the **Shapley value** is in the **core**
 - the core is a **convex** set
 - Shapley value is a **convex combination** of outcomes in the core

Checking Non-emptiness of the Core: Superadditive Games

- An outcome in the core of a superadditive game satisfies the following constraints:
 - $x_i \geq 0$ for all $i \in N$
 - $\sum_{i \in N} x_i = v(N)$
 - $\sum_{i \in C} x_i \geq v(C)$ for any $C \subseteq N$
- A linear feasibility program, with one constraint for each coalition: $2^n + n + 1$ constraints
 - sometimes can be solved in polynomial time solvers using **separation oracles**

Superadditive Games: Computing the Least Core

- LFP for the core LP for the least core
- **min** ϵ
 - $x_i \geq 0$ for all $i \in N$
 - $\sum_{i \in N} x_i = v(N)$
 - $\sum_{i \in C} x_i \geq v(C) - \epsilon$ for any $C \subseteq N$
- A **minimization** program, rather than a **feasibility** program
 - sometimes can be solved in polynomial time using a separation oracle

Superadditive Games: Computing the Cost of Stability

- LFP for the cost of stability:
 $\min \varepsilon$
 $x_i \geq 0$ for all $i \in N$
 $\sum_{i \in N} x_i = v(N) + \varepsilon$
 $\sum_{i \in C} x_i \geq v(C)$ for any $C \subseteq N$
- Observe how ε has moved!
 – sometimes can be solved in polynomial time using a separation oracle

Core and Related Concepts: Non-Superadditive Games

- What if the game is not superadditive?
- Can solve a similar LFP for each coalition structure $CS = (C^1, \dots, C^k)$:
 $x_i \geq 0$ for all $i \in N$
 $\sum_{i \in C^1} x_i = v(C^1)$
 $\dots$
 $\sum_{i \in C^k} x_i = v(C^k)$
 $\sum_{i \in C} x_i \geq v(C)$ for any $C \subseteq N$
- Running time: # of partitions of N x time to solve an exp-sized LFP - infeasible in general.

Complexity Upper Bounds

- Suppose the characteristic function v is poly-time computable
- Deciding whether an outcome is in the core is in coNP:
 – to show that an outcome is not stable, it suffices to guess a coalition that can benefit from deviating
- [Greco et al.'09]: Deciding whether an outcome is
 – in the kernel: in Δ_2^P
 – in the bargaining set: in Π_2^P
 – the nucleolus: in Δ_2^P
- There are matching hardness results for specific representation languages

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Weighted Voting Games

- n parties in the parliament
- Party i has w_i representatives
- A coalition of parties can form a government only if its total size is at least q
 - usually $q \geq \lfloor \sum_{i=1, \dots, n} w_i / 2 \rfloor + 1$: strict majority
- Notation: $w(C) = \sum_{i \in C} w_i$
- This setting can be described by a game $G = (N, v)$, where
 - $N = \{1, \dots, n\}$
 - $v(C) = 1$ if $w(C) \geq q$ and $v(C) = 0$ otherwise
- Observe that weighted voting games are simple games
- Notation: $G = [q; w_1, \dots, w_n]$
 - q is called the **quota**

Weighted Voting Games: UK

- United Kingdom, 2005:
 - 650 seats, $q = 326$
 - **Conservatives (C)**: 196
 - **Labour (L)**: 354
 - **Liberal Democrats (LD)**: 62
 - 8 **other** parties (**O**), with a total of 38 seats
- $N = \{C, L, LD, O\}$
- for any $X \subseteq N$, $v(X) = 1$ if and only if $L \in X$
- **L** is a veto player, **C**, **LD**, and **O** are dummies
- $\phi_L = 1$, $\phi_C = \phi_{LD} = \phi_O = 0$



Weighted Voting Games: UK

- United Kingdom, 2010:
 - 650 seats, $q = 326$
 - **Conservatives (C)**: 307
 - **Labour (L)**: 258
 - **Liberal Democrats (LD)**: 57
 - 8 **other** parties (**O**), with a total of 28 seats
- $N = \{C, L, LD, O\}$
- $v(\{C, L\}) = v(\{C, LD\}) = v(\{C, O\}) = 1$
- $v(\{L, LD\}) = v(\{L, O\}) = v(\{LD, O\}) = 0$, $v(\{L, LD, O\}) = 1$
- **L**, **LD** and **O** are symmetric
- $\phi_C = 1/2$, $\phi_L = \phi_{LD} = \phi_O = 1/6$



Weighted Voting Games as Resource Allocation Games

- Each agent i has a certain amount of a **resource** w_i
 - time or money or battery power
- One or more tasks with a **resource requirement** q and a value V
- If a coalition has enough resources to complete the task (q or more units), it earns its value V , else it earns 0
 - By **normalization**, can assume $V = 1$
- If $q < \sum_i w_i / 2$, grand coalition need not form
 - weighted voting games with **coalition structures**

Shapley Value in Weighted Voting Games

- In a simple game $G = (N, v)$, a player i is said to be **pivotal**
 - for a coalition $C \subseteq N$ if $v(C) = 0$, $v(C \cup \{i\}) = 1$
 - for a permutation $\pi \in P(N)$ if he is pivotal for $S_\pi(i)$
- In simple games player i 's Shapley value = $\Pr[i \text{ is pivotal for a random permutation}]$
 - measure of **voting power**
- Shapley value is widely used to measure power in various voting bodies
- UK elections'10 illustrate that **power $\neq$ weight**

Shapley Value Paradoxes in Weighted Voting Games (1/2)

- Shapley value may sometimes behave in a **counterintuitive** way as the game changes
- Paradox of **new members** [Felsenthal and Machover'98]:
when a new player joins,
the power of an existing player may go **up**
 - $G = [4; 2, 2, 1]$:
player 3 is a dummy, so $\phi_3 = 0$
 - $G = [4; 2, 2, 1, 1]$ (a new player of weight 1 joins):
 $\phi_3 = 1/12$

Shapley Value Paradoxes in Weighted Voting Games (2/2)

- Shapley value may sometimes behave in a **counterintuitive** way as the game changes
- Paradox of **size** [Felsenthal and Machover'98, Aziz et al.'11]:
an agent can **increase** his total power by splitting his weight between two identities
 - $G = [4; 2, 2]$:
players 1 and 2 are symmetric, so $\phi_1 = 1$
 - $G = [4, 1, 1, 2]$
(player 1 splits into two players 1' and 1''):
all players are symmetric, so $\phi_{1'} = \phi_{1''} = \phi_2 = 1/3$

Weighted Voting Games: Computational Aspects

- Deciding if a player is a dummy: **coNP**-complete
- Computing Shapley value and Banzhaf index:
 - **#P**-complete [Deng & Papadimitriou'94]
 - **hard** to approximate
- Computing the core/checking if an outcome is in the core:
 - **poly-time** (since WVG are simple games)
 - if we allow coalition structures, these problems become computationally **hard** [Elkind et al.'08b]

Weighted Voting Games: Computational Aspects

- Computing the value of the least core:
 - NP-hard, but admits an FPTAS [Elkind et al.'09a]
- Computing the cost of stability:
 - NP-hard, but admits an FPTAS [Bachrach et al.'09]
- Computing the nucleolus: NP-hard [Elkind et al.'09a]

Weighted Voting Games: Small Weights

- Suppose all weights are at most polynomial in n
 - realistic in many applications
- Then
 - Shapley value and Banzhaf index can be computed in poly-time by dynamic programming [Matsui & Matsui'00]
 - value of the least core and CoS are poly-time computable [Elkind et al.'09a, Bachrach et al.'09]
 - nucleolus is poly-time computable [Elkind and Pasechnik'09]

WVG and Simple Games

- WVGs are simple games
- Can every simple game be represented as a WVG?
- $G = (N, v)$:
 - $N = \{1, 2, 3, 4\}$
 - $v(C) = 1$ iff $C \cap \{1, 3\} \neq \emptyset$ and $C \cap \{2, 4\} \neq \emptyset$
- Suppose $G = [q; w_1, w_2, w_3, w_4]$

$$\left. \begin{array}{l} w_1 + w_2 \geq q, \quad w_3 + w_4 \geq q \\ w_1 + w_2 + w_3 + w_4 \geq 2q \\ w_1 + w_3 < q, \quad w_2 + w_4 < q \\ w_1 + w_2 + w_3 + w_4 < 2q \end{array} \right\} \text{ a contradiction!}$$

A Generalization: Vector Weighted Voting Games

- The game in the previous slide can be thought of as a combination of two WVGs:
 - $G^{\text{odd}} = [1; 1, 0, 1, 0]$ and $G^{\text{even}} = [1; 0, 1, 0, 1]$
 - to win, a coalition needs to win in both games
- Definition: a k -weighted voting game is a tuple $[N; \mathbf{g}; \underline{w}_1, \dots, \underline{w}_n]$, where $|N| = n$ and
 - $\mathbf{g} = (q^1, \dots, q^k)$ is a vector of k real quotas
 - for each $i \in N$, $\underline{w}_i = (w_i^1, \dots, w_i^k)$ is a vector of k real weights
- $v(C) = 1$ if $\sum_{i \in C} w_i^j \geq q^j$ for each $j = 1, \dots, k$ and $v(C) = 0$ otherwise

Vector Weighted Voting Games

- Given a k-VWVG $G = [N; \mathbf{q}; \mathbf{w}_1, \dots, \mathbf{w}_n]$, we can define $G^j = [q^j; w_1^j, \dots, w_n^j]$
- G^j is a weighted voting game
 - we will refer to G^j as the j -th **component** of G
- To win in G , a coalition needs to win in **each** of the component games
 - we can write $G = G^1 \wedge \dots \wedge G^k$
 - thus, G is a **conjunction** of its component games
- a **k**-VWG models a resource allocation games with **k types of resources**
 - each task needs q^j units of resource j

VWVG in the Wild: EU Voting

- Voting in the European Union is a 3-WVG
 $G = G^1 \wedge G^2 \wedge G^3$, where
 - G^1 corresponds to commissioners
 - G^2 corresponds to countries
 - G^3 corresponds to population
- The players are the 27 member states:
 Germany, UK, France, Italy, Spain, Poland, Romania, The Netherlands, Greece, Czech Republic, Belgium, Hungary, Portugal, Sweden, Bulgaria, Austria, Slovak Republic, Denmark, Finland, Ireland, Lithuania, Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta.



EU Voting Game

- $G^1 = [255; 29, 29, 29, 29, 27, 27, 14, 13, 12, 12, 12, 12, 10, 10, 10, 7, 7, 7, 7, 7, 4, 4, 4, 4, 3]$
- $G^2 = [14; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
- $G^3 = [620; 170, 123, 122, 120, 82, 80, 47, 33, 22, 21, 21, 21, 21, 18, 17, 17, 11, 11, 11, 8, 8, 5, 4, 3, 2, 1, 1]$
 - UK, Greece, Estonia
- For a proposal to pass, it needs to be supported by
 - 74% of the commissioners
 - 50% of the member states
 - 62% of the EU population

VWVGs and Simple Games

- VWVGs are **strictly more expressive** than WVGs
- Theorem:** any simple game can be represented as a **vector** weighted voting game
- Proof: consider a simple game $G = (N, v)$
 - for each losing coalition $C \subseteq N$, we construct a game $G^C = [q^C; w_1^C, \dots, w_n^C]$ as follows:
 - $q^C = 1, w_i^C = 1$ if $i \notin C$ and $w_i^C = 0$ if $i \in C$
 - D loses in G^C iff $D \subseteq C$
 - Let $G^* = \bigwedge_{v(C)=0} G^C$
 - if $v(D) = 0$, D loses in G^D and hence in G^*
 - if $v(D) = 1$, by monotonicity D wins in each component game and hence in G^*

Dimensionality

- Vector weighted voting games form a **complete** representation language for simple games
- However, the construction in the previous slide may use **exponentially many** component games
- **Definition:** the **dimension** $\dim(G)$ of a simple game G is the minimum number of components in its VWVG representation
 - every simple game has dimension $O(2^n)$
 - there exist simple games of dimension $\Omega(2^{n/2-1})$

Dimensionality: Computational Aspects

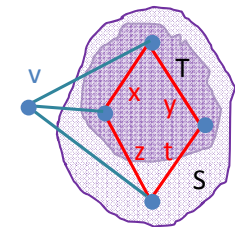
- Suppose we are given a k -VWG G
- $\dim(G) \leq k$; but is $\dim(G) = k$?
 - **NP-hard** to decide, even if $k = 2$ [Deineko & Woeginger'06, Elkind et. al.'08a]
- A seemingly easier question is to decide whether the given representation is **minimal**, i.e., none of the component games is **redundant**
 - **NP-hard** even if
 - ... all weights are in $\{0, 1\}$ (but k is large)
 - ... $k = 2$ (but weights are large)
 - **poly-time** solvable if all weights are polynomial in n and k is bounded by a constant [Elkind et al.'08a]

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Induced Subgraph Games

- Players are vertices of a weighted graph
- Value of a coalition = total **weight of internal edges**
 - $v(T) = x+y$, $v(S) = x+y+z+t$
- Models social networks
 - Facebook, LinkedIn
 - cell phone companies with free in-network calls
- If all edge weights are **non-negative**, this game is convex:
 - $\delta_v(S) \geq \delta_v(T)$



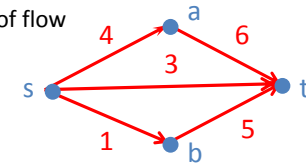
Induced Subgraph Games: Complexity [Deng, Papadimitriou'94]

- If all edge weights are **non-negative**, the core is non-empty
 - also, we can check in **poly-time** if a given outcome is in the core
- In general, determining emptiness of the core is **NP-complete**
- Shapley value is **easy** to compute:
 - let $E = \{e^1, \dots, e^k\}$ be the list of edges of the graph
 - let G^j be the induced subgraph game on the graph that contains edge e^j only
 - we have $G = G^1 + \dots + G^k$
 - $\phi_i(G^j) = w(e^j)/2$ if e^j is adjacent to i and 0 otherwise
 - $\phi_i(G) = (\text{weight of edges adjacent to } i)/2$



Network Flow Games

- Agents are edges in a network with source s and sink t
 - edge e_i has capacity c_i
- Value of a coalition = amount of s – t flow it can carry
 - $v(\{sa, at\}) = 4$, $v(\{sa, at, st\}) = 7$
- Thresholded network flow games (TNFG): there exists a threshold T such that
 - $v(C) = 1$ if C can carry $\geq T$ units of flow
 - $v(C) = 0$ otherwise
- TNFG with $T = 6$
 - $v(\{sa, at\}) = 0$, $v(\{sa, at, st\}) = 1$

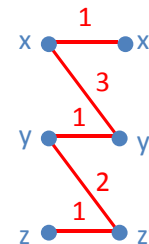


Network Flow Games: Complexity

- Complexity of checking if an agent is a dummy:
 - **coNP-complete**
- Computing Banzhaf power index in TNFG:
 - **#P-complete** [Bachrach&Rosenschein'07]
- Checking if an outcome is in the core/computing the nucleolus in NFG
 - general capacities: **NP-hard**
 - unit capacities: **poly-time** [Deng et al.'06]
- Computing the cost of stability in TNFG:
 - **NP-hard** [Resnick et al.'09]

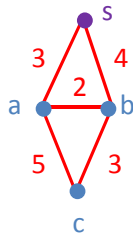
Assignment Games [Shapley & Shubik'72]

- Players are vertices of a bipartite graph (V, W, E)
- Value of a coalition = weight of the max-weight induced matching
 - $v(\{x, y, z\}) = 0$, $v(\{x, x', y'\}) = 3$
- Computing the core/nucleolus:
 - **poly-time** [Solymosi & Raghavan'94]
- Generalization: matching games
 - same definition, but the graph need not be bipartite
 - least core: **poly-time** [Faigle et al.'06]
 - nucleolus: **poly-time** for unit weights [Kern&Paulusma'03]



Cost-Sharing Games

- So far, we assumed that coalitions form to **earn money**
- Alternatively, they can form to **share costs**
 - $v(C) \leq 0$ for all $C \subseteq N$
- Definitions of superadditivity, convexity, etc. need to be modified
- Example: minimum spanning tree games
 - players are vertices of a weighted graph with a distinguished vertex s
 - all players want to connect to s
 - edge weights represent **costs**
 - $v(C) = -w(\text{minimum spanning tree on } C \cup \{s\})$
 - $v(\{a\}) = -3$, $v(\{a, b\}) = -5$, $v(\{a, b, c\}) = -8$



Minimum Spanning Tree Games: Complexity

- The core is always non-empty: build a minimum spanning tree and charge each agent the cost of the upstream link [Granot&Huberman'81]
- Yet, checking if an outcome is in the core is **coNP-complete** [Faigle et al.'97]
- Computing the value of the least core and the nucleolus is **NP-hard** [Faigle et al.'98]

Tutorial Overview

- Introduction
- Definitions
- Solution concepts
- Representations and computational issues
 - oracle representation
 - combinatorial optimization games
 - weighted voting games
 - **complete representation languages**
- Coalition formation in multiagent systems research
- Bayesian models of coalitional games
- Other topics of interest

Coalitional Skill Games [Bachrach&Rosenschein'08]

- Set of **skills** $S = \{s_1, \dots, s_k\}$
- Set of agents N : agent i has a subset of skills $S_i \subseteq S$
- Set of **tasks** $T = \{t_1, \dots, t_m\}$
 - each task t_j requires a **subset of skills** $S(t_j) \subseteq S$
- A **skill set** of a coalition C : $s(C) = \bigcup_{i \in C} S_i$
- Tasks that C can **perform**: $T(C) = \{t_j \mid S(t_j) \subseteq s(C)\}$
- **Utility** function $u : 2^T \rightarrow \mathbb{R}$
 - e.g., **sum** or **max** of values of individual tasks
- Characteristic function: $v(C) = u(T(C))$

Coalitional Skill Games: Expressiveness and Complexity

- Any **monotone** game can be expressed as a CSG:
 - given a game $G = (N, v)$, we create a task t^C and set $u(t^C) = v(C)$ for any $C \subseteq N$
 - each agent i has a unique skill s_i
 - t^C requires the skills of all agents in C
 - set $u(T') = \max \{ u(t) \mid t \in T' \}$
 - $u(T(C)) = \max \{ u(t^D) \mid D \subseteq C \} = \max \{ v(D) \mid D \subseteq C \} = v(C)$
- However, the representation is only **succinct** when the game is naturally defined via a **small set of tasks**
- [Bachrach&Rosenschein'08] discuss complexity of many solution concepts under this formalism

Synergy Coalition Games [Conitzer & Sandholm'06]

- Superadditive game: $v(C \cup D) \geq v(C) + v(D)$ for any two disjoint coalitions C and D
- Idea**: if a game is superadditive, and $v(C) = v(C_1) + \dots + v(C_k)$ for any partition $(C_1, \dots, C_k)$ of C (**no synergy**), no need to store $v(C)$
- Representation: list $v(\{1\}), \dots, v(\{n\})$ and all **synergies**
- Succinct** when there are few synergies
- This representation allows for **efficient checking** if an outcome is in the core.
- However, it is still **hard** to check if the core is non-empty.

Marginal Contribution Nets [leong&Shoham'05]

- Idea**: represent the game by a set of **rules** of the form **pattern $\rightarrow$ value**
 - pattern** is a Boolean formula over N
 - value** is a number
- A rule **applies** to a coalition if its fits the pattern
- $v(C) =$ **sum** of values of all rules that apply to C
- Example:
 - $R_1: (1 \wedge 2) \vee 5 \rightarrow 3$
 - $R_2: 2 \wedge 3 \rightarrow -2$
 - $v(\{1, 2\}) = 3, v(\{2, 3\}) = -2, v(\{1, 2, 3\}) = 1$

Marginal Contribution Nets

- Computing the Shapley value:
 - let $G(R_1, \dots, R_k)$ be the game given by the set of rules $R_1, \dots, R_k$
 - we have $G(R_1, \dots, R_k) = G(R_1) + \dots + G(R_k)$
 - thus, by **additivity** it suffices to compute players' Shapley values in games with a **single rule** R
 - if $R = \psi \rightarrow x$, where ψ is a **conjunction** of k variables, then $\phi_i = x/k$ if i appears in ψ and 0 otherwise
 - a more complicated (but still poly-time) algorithm for **read-once formulas** [Elkind et al.'09b]
 - NP-hard** for if ψ is an **arbitrary** Boolean formula
- Core-related questions are computationally hard [leong&Shoham'05]

Qualitative Coalitional Games (Wooldridge&Dunne'04)

- Often not interested in utilities, but in **goals** – either the goal is satisfied or not
- **QCGs** are a type of coalitional game in which each agent has a set of goals, and wants one of them to be achieved (doesn't care which)
Agents cooperate in QCGs to achieve mutually satisfying sets of goals.
Coalitions have **sets of choices** representing the different ways they could cooperate
Each choice is a set of goals.

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Feasible/Satisfying Goal Sets

- Goal set $G' \subseteq G$ **satisfies** an agent i if $G_i \cap G' \neq \emptyset$.
Goal set $G' \subseteq G$ satisfies a coalition $C \subseteq Ag$ if

$$\forall i \in C, G_i \cap G' \neq \emptyset$$

- A goal set G' is **feasible** for C if $G' \in V(C)$.

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QCGs

A **Qualitative Coalitional Game** (QCG) is a structure:

$$\Gamma = \langle G, Ag, G_1, \dots, G_n, V \rangle$$

where

- $G = \{g_1, \dots, g_m\}$ is a set of **possible goals**;
- $Ag = \{1, \dots, n\}$ is a set of **agents**;
- $G_i \subseteq G$ is a set of goals for each agent $i \in Ag$, the intended interpretation being that any of G_i would satisfy i ;
- $V : 2^{Ag} \rightarrow 2^{2^G}$ is a **characteristic function**, which for every coalition $C \subseteq Ag$ determines a set $V(C)$ of **choices**, the intended interpretation being that if $G' \in V(C)$, then one of the choices available to coalition C is to bring about **all** the goals in G' simultaneously.

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Representing QCGs

- So, how do we represent the function $V : 2^{Ag} \rightarrow 2^{2^G}$?
- We use a formula Ψ_V of propositional logic over propositional variables Ag, G , such that:

$$\Psi[C, G'] = \top \text{ if and only if } G' \in V(C)$$

- “Often” permits **succinct** representations of V .
- Note that given Ψ_V, C, G' , determining whether $G' \in V(C)$ can be done in time polynomial in size of C, G', Ψ_V .

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Fourteen QCG Decision Problems

Problem	Description	Complexity	Q^{mono}
SC	SUCCESSFUL COALITION	NP-complete	NP-complete
SSC	SELFISH SUCCESSFUL COALITION	NP-complete	NP-complete
UGS	UNATTAINABLE GOAL SET	NP-complete	NP-complete
MC	MINIMAL COALITION	coNP-complete	coNP-complete
CM	CORE MEMBERSHIP	coNP-complete	coNP-complete
CNE	CORE NON-EMPTYNESS	D^P -complete	D^P -complete
VP	VETO PLAYER	coNP-complete	-
MD	MUTUAL DEPENDENCE	coNP-complete	-
GR	GOAL REALISABILITY	NP-complete	P
NG	NECESSARY GOAL	coNP-complete	-
EG	EMPTY GAME	coNP-complete	coNP-complete
TG	TRIVIAL GAME	Π_2^P -complete	Π_2^P -complete
GU	GLOBAL UNATTAINABILITY	Σ_2^P -complete	NP
IG	INCOMPLETE GAME	D_2^P -complete	-

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Coalitional Resource Games (CRGs) (Wooldridge&Dunne'06)

- Problem:
where does characteristic function come from?
- One answer provided by Coalitional Resource Games (CRGs).
- Key ideas:
 - achieving a goal requires expenditure of resources;
 - each agent endowed with a profile of resources;
 - coalitions form to pool resource so as to achieve mutually satisfactory set of goals.
- More general than Coalitional Skill Games of Bachrach&Rosenschein'08.

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CRGs

A coalitional resource game Γ is an $(n + 5)$ -tuple:

$$\Gamma = \langle Ag, G, R, G_1, \dots, G_n, \text{en}, \text{req} \rangle$$

where:

- $Ag = \{a_1, \dots, a_n\}$ is a set of agents;
- $G = \{g_1, \dots, g_m\}$ is a set of possible goals;
- $R = \{r_1, \dots, r_t\}$ is a set of resources;
- for each $i \in Ag$, $G_i \subseteq G$ is a set of goals, as in QCGs;
- $\text{en} : Ag \times R \rightarrow \mathbb{N}$ is an endowment function,
- $\text{req} : G \times R \rightarrow \mathbb{N}$ is a requirement function.

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Nine Decision Problems for CRGs

Problem	Complexity
SUCCESSFUL COALITION	NP-complete
MAXIMAL COALITION	coNP-complete
NECESSARY RESOURCE	coNP-complete
STRICTLY NECESSARY RESOURCE	D^P -complete
(C, G', r) -OPTIMAL	NP-complete
R -PARETO OPTIMALITY	coNP-complete
SUCCESSFUL COALITION WITH RESOURCE BOUNDS	NP-complete
CONFLICTING COALITIONS	coNP-complete
ACHIEVABLE GOAL SET	in P

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QCG and CRG Equivalence

- We can define a notion of “equivalence” ($\equiv$) between QCGs and CRGs:
 $\Gamma_1 \equiv \Gamma_2$ means that QCG Γ_1 and CRG Γ_2 agree on the goal sets that are feasible for coalitions
- Given a QCG Γ_1 and CRG Γ_2 , the problem of determining whether $\Gamma_1 \equiv \Gamma_2$ is coNP-complete.

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Can we translate between QCGs and CRGs?

Four questions suggest themselves:

- 1 Given a CRG, Γ , is there always a QCG, Q_Γ such that $Q_\Gamma \equiv \Gamma$?
- 2 Given a QCG, Q , is there always a CRG, Γ_Q such that $\Gamma_Q \equiv Q$?
- 3 How “efficiently” can a given CRG be expressed as an equivalent QCG in those cases where such an equivalent structure exists?
- 4 How “efficiently” can a given QCG be expressed as an equivalent CRG in those cases where such an equivalent structure exists?

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Translating CRGs $\rightarrow$ QCGs

- We can always translate a CRG into an equivalent QCG.
- More interestingly, we can do this **efficiently**:
for every CRG Γ_1 there exists an equivalent QCG Γ_2 such that $|\Gamma_2| \leq |\Gamma_1|^2$.

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Translating QCGs to CRGs

- We **cannot** always translate QCGs to equivalent CRGs.
- Moreover, even when we **can** translate, we can't always do it efficiently:
*there exist QCGs Γ for which equivalent CRGs exist but for which the size of the **smallest** equivalent CRG is at least $2^{|\Gamma|}$*

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- 1 Introduction
- 2 Definitions
- 3 Solution concepts
- 4 Computational issues
- 5 Examples of coalitional games
- 6 Coalition formation in multiagent systems research
 - 1 Coalition structure generation algorithms
 - 2 Coalition formation algorithms
 - through bargaining / negotiations / auctions
 - in the economics and in the multiagent systems literature
 - with complete or incomplete information (under uncertainty)
- 7 “Bayesian” models of cooperative games
- 8 Other topics of interest

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Why Coalitions

- Task Allocation
 - Resource allocation
 - Complementary agent expertise



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- Cooperative game theory / coalitional games inspire Multiagent Systems Research
- **Multiagent Systems:** give rise to natural paradigms of **Coalition Formation**
 - Need to form teams of intelligent, selfish agents to serve tasks
 - Task execution results to utility (value/payoff) to teams
 - Utility and resource consumption have to be distributed among members

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Why Coalitions

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Why Coalitions

- Task Allocation
 - Resource allocation
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Why not just Game Theory

- Well, agents are not intelligent or “rational”!
- Most times, a static view – “how” not addressed
- Stability concepts not adequate for AI
 - e.g., coalition *structure* stability
- Dynamic formation equilibria : next to *impossible to compute*
- *Uncertainty* is usually disregarded
- Assumptions are restrictive
 - e.g., agents can belong to multiple coalitions
- Other information is usually disregarded
 - e.g., environments might have structure

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Coalition Formation in Multiagent Systems++

- Questions / Issues / Opportunities:
 - **Coalitional Structure Generation** : “How to achieve maximum social welfare, and serve multiple given tasks?”
 - disregarding allocation issues
 - **How to form coalitions dynamically? How to distribute value / resources - in a decentralized way, if possible?**
 - Coalition formation protocols / bargaining / auction approaches to achieve this
 - **Uncertainty** is naturally of great interest
 - This resulted to contributions in *coalition formation algorithms under uncertainty*, and research on *stability under uncertainty*
 - Links to **Mechanism Design**:
 - How to maximize welfare, and incentivize individually rational agents to **truthfully reveal** their preferences to a center?
 - ...and assure stable coalitions?

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Coalition Structure Generation

This line of research typically tries to **partition** the space into disjoint coalitions, so that to

find the “optimal coalition structure”

that is, the structure with the greatest overall value

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Coalition Structure Generation Methods

- **Dynamic programming** (Yeh 1986, Rothkopf *et al.* 1995, Rahwan and Jennings 2008)
 - Exploits “optimal substructures” of the problem: an optimal solution can be constructed from optimal solutions of subproblems
 - For every coalition, it evaluates the possible ways of splitting it in two, and stores the best possible split in a table. This information is then used to obtain the optimal structure.
 - Finds the optimal solution in $O(3^n)$, but is not anytime

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- **Heuristics** (Shehory and Kraus 1998, Sen and Dutta 2000)
 - Examples include using genetic algorithms, or restricting the sizes of the coalitions that are taken into consideration
 - Find “good” solutions quickly, but provide no guarantees

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Coalition Structure Generation Methods

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 - Examples include using genetic algorithms, or restricting the sizes of the coalitions that are taken into consideration
 - Find “good” solutions quickly, but provide no guarantees
- **Anytime Algorithms**
 - Return an initial solution quickly, then progressively find better solutions until optimal
 - Provide worst-case guarantees, but to establish an n -bound at least 2^{n-1} structures have to be searched

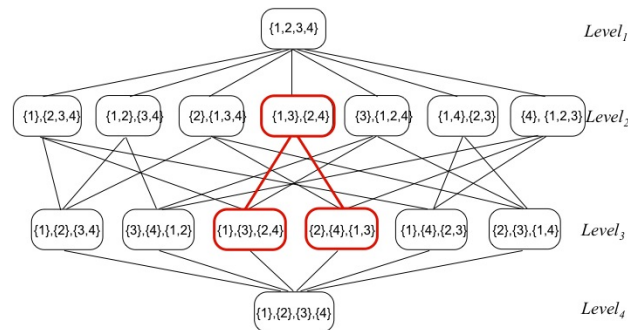
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Anytime Coalition Structure Generation

- (Sandholm, Larson, Andersson, Shehory and Tohme, 1999)
 - They use a particular representation of the search space called the “coalition structure graph”

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Anytime Coalition Structure Generation



- Coalition structure graph for a 4-agent game. Nodes represent coalition structures. Arcs represent mergers of two coalitions when followed downward, and splits of a coalition into two when followed upward.

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Anytime Coalition Structure Generation

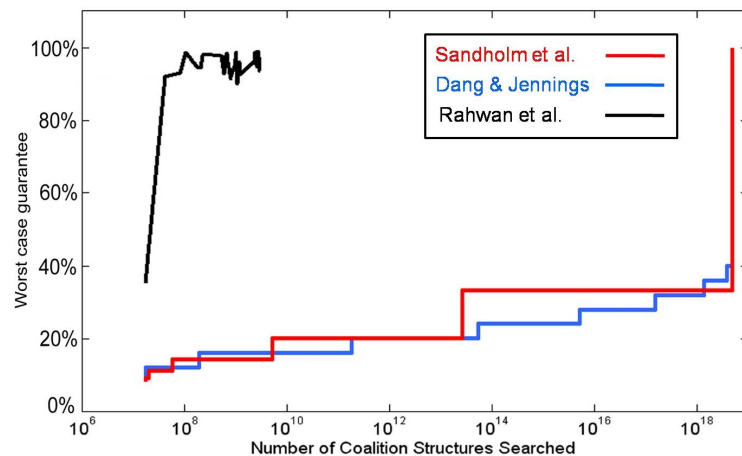
- (Sandholm, Larson, Andersson, Shehory and Tohme, 1999)
 - They use a particular representation of the search space called the “coalition structure graph”
 - They prove that, after searching particular levels of the graph, the solution is within a certain bound from the optimal
- (Dang and Jennings, 2004)
 - Also use the coalition structure graph, but...
 - Instead of partitioning it into levels, they define particular subsets of the graph
 - After a subset is searched, the solution is within a certain bound
- (Rahwan and Jennings, 2007)
 - Use novel representation for the search space’s subspaces
 - Each subspace is a collection of coalition structures
 - Each subspace uniquely corresponds to an integer partition, e.g., $P_{\{1,1,2\}}$, with indices (parts) being the sizes of coalitions in the structure
 - Use branch-and-bound techniques to search each sub-space

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Results with 25 agents

The worst case guarantee

(as a function of the No. of coalition structures searched)



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Formation/Bargaining with Complete Information

(Chatterjee, Dutta and Segupta, Review of Economic Studies-1993)

- Discounted coalitional bargaining model with a *fixed proposer order*
- Unlike (Rubinstein, 1982) on alternating offers of 2-person and n -person unanimus bargaining ...
- Chatterjee *et al.*: **delay of agreement can occur in n -person coalitional bargaining**
- ...in (a stationary) *subgame-perfect* equilibrium
- environment is *superadditive*

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Formation/Bargaining with Complete Information++

(Akira Okada, *Games and Economic Behavior*-1996)

- Discounted coalitional bargaining model with *random proposers*
- *Unlike* Chatterjee *et al.*, ...
- Okada: **no delay of agreement in equilibrium**
- ...that is, in (a stationary) *subgame-perfect* equilibrium
- environment is, again, *superadditive*

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Dynamic Coalition Formation

- Formation process is viewed as a Markov chain, and its properties are studied
- Example: (Dieckmann and Schwalbe, 1998)
 - Formation: A *dynamic process* model with random proposers and specific rules:
 - join an existing coalition, suggesting a specific allocation vector
 - suggest a modification of the allocation vector in his current coalition
 - form a singleton
 - Agents play “**best responses**” : they are myopic maximizers and choose to move to coalitions that can guarantee them the highest feasible payoff given current demands
 - The process has at least one *absorbing state*
 - Obviously, **core elements** ($\langle$ coalition structure, allocation vector $\rangle$ pairs), **are absorbing states**

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Dynamic Coalition Formation++

- but the opposite is not true
- Example: $N = \{1, 2, 3\}$, $v(\{i\}) = 2 \forall i \in N$, $v(S) = 5$ for $|S| = 2$, and $v(N) = 8$
 - state $\omega = \langle \{N\}, \{4, 2, 2\} \rangle$ is absorbing *given the rules of the game*
 - ...
 - ...but **not in the core** : had coalition $\{2, 3\}$ been able to form, it would have done better
 - i.e., both agents 2 and 3 would prefer to join forces, but are not permitted

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Dynamic Coalition Formation++

- Formation process is viewed as a Markov chain, and its properties are studied
- Example: (Dieckmann and Schwalbe, 1998)
 - Formation: A *dynamic process* model with random proposers and specific rules
 - Agents play “**best responses**” : they are myopic maximizers and choose to move to coalitions that can guarantee them the highest feasible payoff given current demands
 - The process has at least one *absorbing state*
 - If *exploratory* (i.e., “suboptimal”) moves are added to the model then *absorbing states correspond to core elements*, and **convergence to core elements is shown** (if core non-empty)
 - Note: The core is essentially a core *with coalition structures*

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Dynamic Coalition Formation ++

- (Konishi and Ray, 2002) study a somewhat related formation process
 - chain transitions occur only if moves are beneficial to all
 - under specific assumptions, process converges to the “classic” core

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Dynamic Coalition Formation ++

- An early *distributed* approach from the M-A systems community: (Shehory and Kraus, IJCAI-1995)
- Agents approach others and acquire *full information* about their capabilities
 - Stage 1: coalitional values are calculated in a decentralised manner (a protocol determines how agents approach others and commit to calculating values)
 - Calculation is performed given knowledge of tasks and capabilities
 - Stage 2: an iterative greedy process through which the agents decide upon the preferred coalitions and form them.
- Complexity: exponential to agent number
- Heuristics: fix coalitional membership size to small value to reduce communication and calculations costs

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Bilateral Negotiations under Incomplete Information

- (Ketchpel, AAAI-1994)
 - a two-agent auction protocol to facilitate coalition formation in the face of uncertain reward
- (Kraus, Wilkenfeld and Zlotkin, Artificial Intelligence-1995)
 - infinite horizon alternating offers model of bargaining
 - agents need to negotiate about sharing a resource
 - agents update beliefs about each other's capabilities given observed negotiations
 - however, the negotiation in any given period is *bilateral*
 - not more than 2 agents need to share a resource at a time

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Uncertainty about capabilities and/or resources

Formation algorithms that take agent capabilities into account

- (Shehory, Sycara and Jha, 1997) : Coalition Formation in the RETSINA framework (an open MAS that supports communities of heterogeneous agents)
 - communities of peers ; no centralized control ; distributed infrastructural services facilitate the interactions between agents
 - Coalition formation services see that tasks of common interest are executed successfully
 - i.e.: Social welfare maximization is sought after : Does not deal with payoff allocation.
- (Shehory and Kraus, 1998)
 - Does not deal with payoff allocation.
- (Shehory and Kraus, 1999)
 - Does deal with payoff allocation.

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Uncertainty about capabilities and/or resources

- (Shehory, Sycara and Jha, 1997) : Coalition Formation in the RETSINA framework
 - Does not deal with payoff allocation.
- (Shehory and Kraus, 1998)
 - Does not deal with payoff allocation.
- (Shehory and Kraus, 1999)
 - Does deal with payoff allocation.
- Assumption: **Explicit Collaboration / Communication of capabilities-related information**

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Feasible Formation of Coalitions among Autonomous Agents in Non-superadditive Environments (Shehory and Kraus 1999)

- Two coalition formation algorithms for use in non-superadditive environments.
- Algorithms seek to satisfy the **kernel** stability concept
 - One assumes full cooperation/distribution of required calculations (of coalition values and kernel-stable configurations)
 - guarantees a bound from social welfare *assuming a specific coalitional configuration*
 - The second algorithm is negotiation-based ; and is *anytime* w.r.t. ϵ -stability
- The algorithms deal with *expected* payoffs given reported resources and time constraints
- Information about the capabilities and resources of others is obtained via communication, or direct observation of the resources of coalitions and the payoffs they have received

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Making Compromises to Form Coalitions under Incomplete Information

(Kraus, Shehory and Taase AAMAS-2004)

- coalition formation in the Request for Proposal domain
- tasks comprised of subtasks; each subtask should be performed by a different agent
- value/costs of a subtask to another agent unknown, but overall task payoff and agent capabilities are known
- central authority makes the *most rewarding* matches among agents, based on their suggestions / constraints
- **Allocation strategies:**
 - 1 equal
 - 2 proportional to costs
 - 3 kernel-stable
 - 4 any of the above with the use of **compromises**
- Findings: Compromises improve social welfare

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Stochastic Cooperative Games

Suijs, Borm, De Waegenare and Tijs-Eur.J.Econ.Research; Suijs and Borm-Games and Econ. Behavior (1999)

- A more rigorous handling of coalition value uncertainty
- **stochastic cooperative games (SCGs)**: a set of agents, a set of *coalitional actions*, and...
- a function *assigning to each action a random variable with finite expectation* (coalitional payoff given action)
- they introduce *relative* demand vectors (payoff “shares”)
- this is to deal with *expected* payoffs
- Main Results:
 - deal with the allocation of the stochastic payoff of the *grand coalition*
 - core for grand coalition; “balancedness” for SCGs; convexity; core existence under different types of *risk behaviour*
 - ...but no “type uncertainty” ; and players do *not* have *private* beliefs

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Bayesian Cooperative Games

Chalkiadakis and Boutilier (AAMAS-04); Chalkiadakis (2007);
Chalkiadakis *et al.* (IJCAI'07, AAMAS'07)

- Explicit handling of **agent type uncertainty** : introduced “**Bayesian Coalition Formation Problems**” (or Games)
- agents have **private probabilistic beliefs** $B_i(\mathbf{t}_j)$ about the **types (capabilities)** of others;
- coalitions take coalitional actions $a \in A_C$ with *stochastic* outcomes $s \in S$ that occur with $Pr(s|a, \mathbf{t}_C)$ and result to reward $R(s)$
- relative demand vectors are assumed: $\mathbf{d}_C = \langle d_i \rangle$, $\sum_{i \in C} d_i = 1$
- type-uncertainty is cast into coalitional value uncertainty; agents have expectations about the value of any
 - coalition-action pair: $Q_i(C, a) = \sum_{\mathbf{t}_C \in T_C} B_i(\mathbf{t}_C) \sum_s Pr(s|a, \mathbf{t}_C) R(s)$
 - coalition: $V_i(C) = \max_{a \in A_C} Q_i(C, a)$
 - and coalitional agreement $\langle C, a, \mathbf{d}_C \rangle$ to themselves:
 $p_i^j(C, a, \mathbf{d}_C) = \frac{d_j}{\sum_{k \in C} d_k} V_i(C)$; or to others:
 $p_k^j(C, a, \mathbf{d}_C) = \frac{d_k}{\sum_{k \in C} d_k} V_i(C)$

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Bayesian Cooperative Games

- Stability now has to be ensured **given the agents' beliefs**
- Three variants of the **Bayesian Core (BC)** are defined for *coalition structures*
 - **weak BC**: There exists no coalitional configuration $\langle C, \mathbf{d}_C, a_C \rangle$ of agents s.t. each believe they can be better off in $\langle C, \mathbf{d}_C, a_C \rangle$ than in the BC configuration
 - **strict BC**: No agent in a BC configuration believes that there exists a configuration which could provide it with more payoff than it currently expects to receive *and* which others believe not harmful
 - **strong BC**: No agent in a BC coalition believes that there exists an *expected to be feasible* configuration which could provide it with more payoff than it currently expects to receive
 - i.e., the agent assesses its own beliefs about the value of an agreement to its partners
- **strict BC** $\subseteq$ **weak BC** ; and, if agents' beliefs coincide, the strict and the strong BC coincide

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Bayesian Cooperative Games

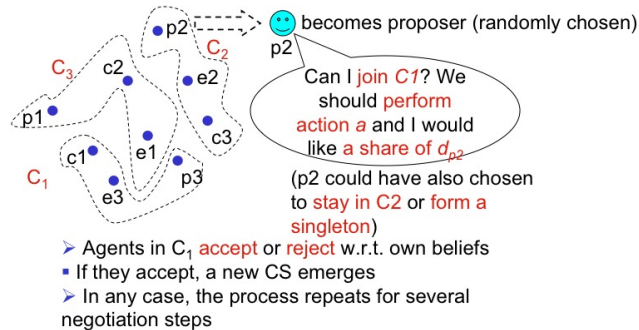
Other results:

- a **dynamic process** with random proposers is defined and proved to converge to the **strong BC**
 - The process uses “best replies with experimentation” (links to Diekmann and Schwalbe's work)
- A heuristic algorithm for coalitional bargaining to approximate **PBE** equilibrium play using belief updating
- links to (Bayesian) reinforcement learning; integration into a framework for *optimal repeated coalition formation under uncertainty*
 - tackles agents' sequential decision making problem in repeated formation environments

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Bayesian Cooperative Games

Dynamic Coalition Formation under Uncertainty



- The process induces a *Markov chain* with states $\langle CS^t, \mathbf{a}_{CS}^t, \mathbf{d}_{CS}^t \rangle$
- **Best Reply** process: best *expectedly feasible* demand w.r.t. beliefs
- **Best Reply with Experimentation (BRE)** : destabilizes coalition structure in order to explore potentially profitable structures (given beliefs)

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Bayesian Cooperative Games

Other results:

- a **dynamic process** with random proposers is defined and proved to converge to the *strong BC*
 - The process uses “best replies with experimentation” (links to Dieckmann and Schwalbe’s work)
- A heuristic algorithm for coalitional bargaining to approximate *PBE* equilibrium play using belief updating
 - Compared with the (Kraus *et al.* 2003, 2004) coalition formation with compromises algorithm
- links to (Bayesian) reinforcement learning; integration into a framework for *optimal repeated coalition formation under uncertainty*
 - tackles agents’ sequential decision making problem in repeated formation environments

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Bayesian RL for Sequentially Optimal Repeated Coalition Formation under Uncertainty

- the *Q*-values now describe the *sequential* value of current coalitional agreements, assuming repeated formation:

$$Q_i(C, a_C, \mathbf{d}_C, B_i) = \sum_s \Pr(s|C, a_C, B_i) [r_i R(s) + \gamma V_i(B_i^{s, a_C})] \quad (1)$$

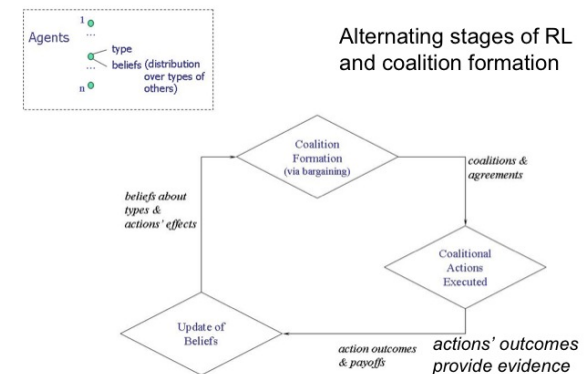
$$= \sum_{\mathbf{t}_C} B_i(\mathbf{t}_C) \sum_s \Pr(s|a_C, \mathbf{t}_C) [r_i R(s) + \gamma V_i(B_i^{s, \alpha})]$$

$$V_i(B_i) = \sum_{C|C \in \mathcal{C}, \mathbf{d}_C} \Pr(C, a_C, \mathbf{d}_C | B_i) Q_i(C, a_C, \mathbf{d}_C, B_i) \quad (2)$$

- belief-state MDP determines long-term value Q_i of any coalitional agreement: how *i*’s beliefs will change in the future when deciding how useful an agreement is now
- tractable Bayesian methods can approximate the MDP solution

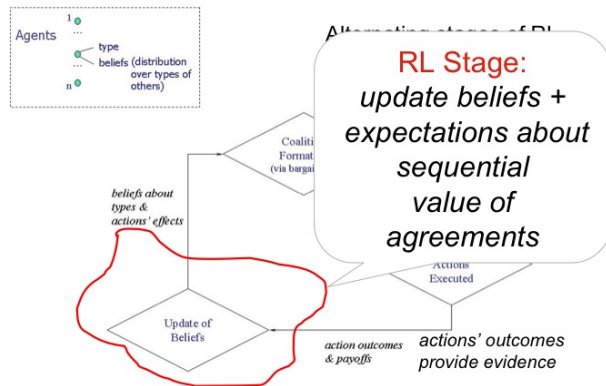
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Bayesian RL for Sequentially Optimal Repeated Coalition Formation under Uncertainty



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Bayesian RL for Repeated Coalition Formation under Uncertainty



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Bayesian Cooperative Games++

leong and Shoham : **Bayesian Coalitional Games** (AAAI-08)

- a related model, with uncertainty *regarding potential worlds*
- agents possess:
 - a common prior over the set of possible coalitional games
 - private information about the true state of the world, captured by their information partitions
- an *ex-interim*, *ex-ante* and an *ex-post* core are defined
- infinite level of mutual modeling in their interim core
- they focus on the stability of the grand coalition
- they show that checking for core emptiness reduces to linear feasibility problems when agents are risk-neutral

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Tutorial Overview

- 1 Introduction
- 2 Definitions
- 3 Solution concepts
- 4 Computational issues
- 5 Examples of coalitional games
- 6 Coalition formation in multiagent systems research
- 7 "Bayesian" models of cooperative games
- 8 **Other topics of interest:**
 - Links between cooperative and non-cooperative solution concepts
 - Cooperative games and mechanism design
 - Cooperative games with "overlapping" coalitions

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Are non-cooperative solution concepts linked to cooperative solution concepts?

Line of research establishes connections between coalitional bargaining equilibrium play outcomes and the core of the underlying coalitional games

- equilibrium payoff sets in particular coalitional bargaining games correspond to core allocations for the participating agents
- deterministic/complete information case: (Perry and Reny, 1994; Moldovanu and Winter, 1995; Hart and Mas-Colell, 1996; Evans, 1997; Serrano and Vohra, 1997; Yan, 2003)
- incomplete information case: (Chalkiadakis, Markakis and Boutilier 2007)

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Are non-cooperative solution concepts linked to cooperative solution concepts? ++

For example, (Moldovanu and Winter, 1995) :

- Introduce **order independent equilibria (OIE)** in coalitional bargaining games:
- A strategy profile is an OIE if, for any specification of first movers in the sequential game, it remains an equilibrium and leads to the same payoff.
- They then show that:
 - Payoffs in OIE that use pure, stationary strategies must be in the core of the underlying game in coalitional form
 - If the underlying game has the property that all its subgames have nonempty cores then, for each payoff vector, there exists an OIE with the same payoff

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Incentive Compatible Coalitional Games

Myerson (1984, 2007)

- a model for **cooperative games with incomplete information** to address mechanism design issues in cooperative games
- the expected utility of an agent is w.r.t. a mechanism that provides him with a sidepayment
- mechanisms assign payoff to agents depending on reported agent types and specified agents' actions
- mechanisms are required to be *truthful*
- the core is defined, but it has to satisfy certain incentive compatibility constraints
 - an outcome is in the core if *not blocked*:
 - a *blocking mechanism* selects a random coalition according to some known distribution, asks agents their types, picks actions, and re-assigns payoffs so that agents are weakly better-off and the mechanism makes a profit in expectation
 - thus, any planned side-payments are required to satisfy incentive compatibility

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Coalitional Games in Open Anonymous Environments

(Yokoo *et al.*, 2005) coined the anonymity-proof core and the core for skills concepts

- However, these concepts do not take into account the uncertainty or beliefs of the agents regarding the type of others
- the agents report their types to a special “mechanism designer” agent, who implements a payoff allocation function that produces an outcome in the core
- the function (the payoff to an agent) depends only on the agent skills and the distribution of skills among other agents
- they show that such a function exists if a set of axiomatic conditions is satisfied (any such function is in the anonymity-proof core); *conditions*: function is
 - 1 anonymous
 - 2 never blocked by any coalition
 - 3 always feasible, always distributes all of the value
 - 4 robust against “hiding skills”
 - 5 robust against “false-name manipulations”
 - 6 robust against collusions

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Coalitional Games in Open Anonymous Environments

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(similar list of authors, 2008) :

- they introduced the “anonymity-proof Shapley value”
- Shapley value for open anonymous environments - Shapley value given skills
- they do not need to compute/store the entire outcome function, only the payoff division for the grand coalition, and compute the rest on demand
- “cheaper” than the anonymity-proof core: $O(2^n)$ to compute, vs. $O(n2^n)$; $O(n)$ storage costs (variables stored) vs. $O(n2^n)$
- like Myerson, they do not deal with any sort of decentralized coalition formation process

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Overlapping Coalition Formation

(Chalkiadakis, Elkind, Markakis and Jennings, 2008)

- Defined **Overlapping Coalition Formation (OCF)** games, and studied stability issues
- partial* or *overlapping* coalition: $\mathbf{r} = \langle r_1, \dots, r_n \rangle$
- $r_i \in [0, 1]$: fraction of i 's resources contributed to $\mathbf{r}$; $r_j = 0$ means j not in coalition
- v maps each $\mathbf{r}$ to payoff: $v : [0, 1]^n \rightarrow \mathbb{R}$
- allocation $\mathbf{x}$ is now a *collection* of payoff vectors
- links to *fuzzy cooperative games*; but the focus is on the stability of *coalition structures*, not just the grand coalition
 - An outcome in the **Aubin core** is a *distribution of the grand coalition's value*, where for each $\mathbf{r}$ the members' payoff is $\geq v(\mathbf{r})$
- the **OCF-core** is defined and *characterized*

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Overlapping Coalition Formation++

- i.e., necessary and sufficient conditions to guarantee that $\langle CS, \mathbf{x} \rangle$ elements or CS structures are in the core are identified
- Given a game (N, v) , where v has some natural properties, an outcome $(CS, \mathbf{x})$ is in the core of (N, v) *iff* for all $S \subseteq N$

$$\sum_{j \in S} p_j(CS, \mathbf{x}) \geq v^*(S). \quad (3)$$

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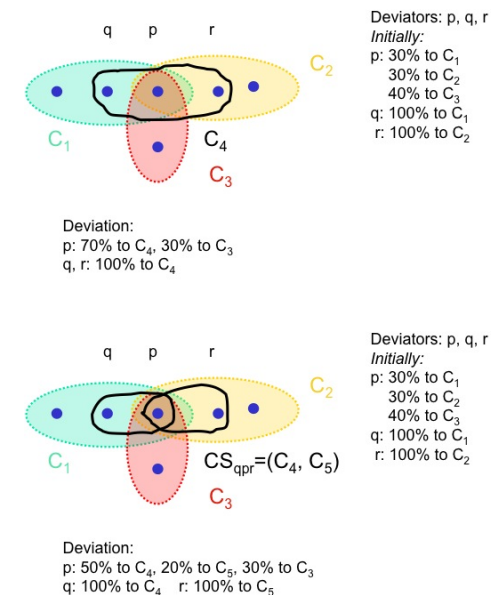
Overlapping Coalition Formation++

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 - An outcome in the **Aubin core** is a *distribution of the grand coalition's value*, where for each $\mathbf{r}$ the members' payoff is $\geq v(\mathbf{r})$
- the **OCF-core** is defined and characterized
- convex* OCF games are defined and studied

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Overlapping Coalition Formation: Deviations



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Some more interesting applications

- **Disaster management**
 - Team formation and cooperation in the ALADDIN project
 - Planning humanitarian relief operations (Pechoucek et al.)
- **Coalitions for power transportation**
 - Firms trade to form coalitions to generate, transmit and distribute electric power (Yeung et al., Contreras et al.)

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Other research directions

- **Coalitions and learning in organizations**
 - Multiple layers of learning; transfer of knowledge among layers
 - Scalability due to structure, hierarchies
- **Safety and robustness**
 - Communication links might fail; what happens to negotiations?
- **Communication costs**
 - How to account for them?
 - Communication complexity of protocols?

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Other research directions++

- **Games with Externalities (Partition Function Games)**
 - What if the value of a coalition depends on *other* formed coalitions, i.e., the coalition structure in place determines the coalitional values?
 - Recent work has focused on efficient PFG representations and coalition structure generation in PFG settings (Michalak et al. 2009; Rahwan et al. 2009).

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References

- [Bachrach and Rosenschein(2007)] Y. Bachrach, J. Rosenschein. Computing the Banzhaf power index in network flow games, In *Proc. AAMAS'07*, 2007.
- [Bachrach and Rosenschein(2008)] Y. Bachrach, J. Rosenschein. Coalitional skill games. In *Proc. AAMAS'08*, 2008.
- [Bachrach et al.(2009)] Y. Bachrach, E. Elkind, R. Meir, D. Pasechnik, M. Zuckerman, J. Rothe, J. Rosenschein. The Cost of stability in coalitional games. In *Proc. SAGT'09*, 2009.
- [Banzhaf(1965)] J. F. Banzhaf. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.
- [Bilbao et al.(2002)] J. M. Bilbao, J. R. Fernández, N. Jiminéz, and J. J. López. Voting power in the European Union enlargement. *European Journal of Operational Research*, 143:181–196, 2002.
- [Bird(1976)] C. G. Bird. On cost allocation for a spanning tree: A game theory approach. *Networks* 6:335–350, 1976.
- [Branzei et al.(2005)] Branzei, R., Dimitrov, D., Tijs, S. Models in cooperative game theory. Springer, 2005.
- [Chalkiadakis and Boutilier(2004)] Chalkiadakis, G., Boutilier, C.. Bayesian reinforcement learning for coalition formation under uncertainty. In *Proc. AAMAS'04*, 2004
- [Chalkiadakis and Boutilier(2007)] Chalkiadakis, G., Boutilier, C. Coalitional bargaining with agent type uncertainty. In *Proc. IJCAI'07*, 2007.
- [Chalkiadakis et al.(2008)] G. Chalkiadakis, E. Elkind, E. Markakis, M. Polukarov, and N. Jennings. Cooperative games with overlapping coalitions. *Journal of Artificial Intelligence Research* 39:179–216, 2010.
- [Chalkiadakis et al.(2007)] Chalkiadakis, G., Markakis, E., Boutilier, C., 2007. Coalition formation under uncertainty: bargaining equilibria and the Bayesian core stability concept. In *proc. AAMAS'07*, 2007.
- [Chatterjee et al.(1993)] Chatterjee, K., Dutta, B., Sengupta, K. A Noncooperative Theory of Coalitional Bargaining. *Review of Economic Studies* 60:463–477, 1993.
- [Conitzer and Sandholm(2003)] Conitzer, V., Sandholm, T., 2003. Complexity of determining non-emptiness of the core. In *Proc. IJCAI'03*, 2003.
- [Conitzer and Sandholm(2006)] V. Conitzer and T. Sandholm. Complexity of constructing solutions in the core based on synergies among coalitions. *Artif. Intell.*, 170:607–619, 2006.

- [Contreras et al.(1998)] Contreras, J.; Klusch, M.; Yen, J., 1998 Multi-Agent Coalition Formation in Power Transmission Planning. Proc. 4. International Conference on Artificial Intelligence in Planning Systems (AIPS-98), June 8-10, 1998, Pittsburgh (USA).
- [Contreras et al.(1999)] Contreras, J.; Wu, F.; Klusch, M.; Shehory, O., 1999 Coalition formation in a power transmission planning environment Proc. of PAAM-97, London, UK, April 1997.
- [Davis and Maschler(1965)] Davis, M., Maschler, M. The Kernel of a Cooperative Game. *Naval Research Logistics Quarterly* 12:223–259, 1965.
- [Deineko and Woeginger(2006)] V. G. Deineko and G. Woeginger. On the dimension of simple monotonic games. *European Jnl of Oper. Res.* 170(1):315–318, 2006.
- [Deng and Papadimitriou(1994)] X. Deng and C. H. Papadimitriou. On the complexity of cooperative solution concepts. *Math. of Oper. Res.*, 19(2):257–266, 1994.
- [Deng et al.(2006)] X. Deng, Q. Fang, and X. Sun. Finding nucleolus of flow game. In *Proc. SODA'06*, 2006.
- [Dieckmann and Schwalbe(1998)] Dieckmann, T., Schwalbe, U. Dynamic Coalition Formation and the Core. Economics Department Working Paper Series, Department of Economics, National University of Ireland - Maynooth, 1998.
- [Elkind et al.(2009a)] E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. Computational complexity of weighted voting games. *Ann. Math. Artif. Intell.*, 56(2):109–131, 2009.
- [Elkind et al.(2008a)] E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. On the dimensionality of voting games. In *Proc. AAAI'08*, 2008.
- [Elkind et al.(2009b)] E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. A tractable and expressive class of marginal contribution nets and its applications *Mathematical Logic Quarterly*, 55(4):362–376, 2009.
- [Elkind et al.(2008b)] E. Elkind, G. Chalkiadakis, N. R. Jennings. Coalition structures in weighted voting games, In *Proc. ECAI'08*, 2008.
- [Elkind and Pasechnik(2009)] E. Elkind, D. Pasechnik. Computing the nucleolus of weighted voting games. In *Proc. SODA'09*, 2009.
- [Evans(1997)] Evans, R. Coalitional bargaining with competition to make offers. *Games and Economic Behavior* 19:211–220, 1997.
- [Faigle et al.(1997)] U. Faigle, W. Kern, S. P. Fekete, W. Hochstättler. On the complexity of testing membership in the core of min-cost spanning tree games. In *Int Jnl of Game Theory*, 26:361–366, 1997.

- [Faigle et al.(1998a)] U. Faigle, W. Kern, J. Kuipers. Computing an element in the lexicographic kernel of a game. In *Math. Meth. Oper. Res.*, 63: 427–433, 2006.
- [Faigle et al.(1998b)] U. Faigle, W. Kern, J. Kuipers. Computing the nucleolus of min-cost spanning tree games is NP-hard. In *Int. Jnl of Game Theory* 27:443–450, 1998
- [Gillies(1953)] Gillies, D. Some Theorems on n-Person Games. Ph.D. thesis, Department of Mathematics, Princeton University, Princeton, 1953.
- [Granot and Huberman(1981)] D. Granot and G. Huberman. Minimum cost spanning tree games. In *Math. Prog.* 21:1–18, 1981.
- [Hart and Mas-Colell(1996)] Hart, S., Mas-Colell, A. Bargaining and Value. *Econometrica* 64(2):357–380, 1996.
- [Jeong and Shoham(2005)] S. Jeong, Y. Shoham. Marginal contribution nets: A compact representation scheme for coalitional games. In *Proc. ACM EC-2005*, 2005.
- [Jeong and Shoham(2008)] Jeong, S., Shoham, Y., 2008. Bayesian coalitional games. In *Proc. AAAI'08*, 2008.
- [Kahan and Rapoport(1984)] Kahan, J., Rapoport, A. Theories of Coalition Formation. Lawrence Erlbaum Associates, 1984.
- [Kern and Paulusma(2003)] W. Kern and D. Paulusma. Matching games: the least core and the nucleolus. *Math. Oper. Res.*, 28(2):294–308, 2003.
- [Ketchpel(1994)] Ketchpel, S. Forming coalitions in the face of uncertain rewards. In *Proc. AAAI'94*, 1994.
- [Klusch and Shehory(1996)] Klusch, M., Shehory, O. Coalition formation among rational information agents. Lecture Notes in Artificial Intelligence (1038), 1996.
- [Konishi and Ray(2002)] Konishi, H., Ray, D. Coalition formation as a dynamic process. Boston College Working Papers in Economics 478, 2002.
- [Kraus et al.(2003)Kraus, Shehory, and Taase] Kraus, S., Shehory, O., Taase, G. Coalition formation with uncertain heterogeneous information. In *Proc. AAMAS'03*, 2003.
- [Kraus et al.(2004)] Kraus, S., Shehory, O., Taase, G. The advantages of compromising in coalition formation with incomplete information. In *Proc. AAMAS'04*, 2004.
- [Kraus et al.(1995)] Kraus, S., Wilkenfeld, J., Zlotkin, G. Multiagent negotiation under time constraints. In *Artificial Intelligence* 75(2), 297–345, 1995.

- [Li et al.(2003)] Li, C., Chawla, S., Rajan, U., Sycara, K., 2003. Mechanisms for coalition formation and cost sharing in an electronic marketplace. In *Proc. ICEC'03*, 2003.
- [Li and Sycara(2002)] Li, C., Sycara, K. Algorithms for combinatorial coalition formation and payoff division in an e-marketplace. In *Proc. AAMAS'02*, 2002.
- [Luce and Raiffa(1957)] Luce, R., Raiffa, H. Games and Decisions. John Wiley & Sons, New York, New York, 1957.
- [Malizia et al.(2007)] E. Malizia, L. Palopoli, F. Scarcello. Infeasibility certificates and the complexity of the core in coalitional games. In *Proc. IJCAI'07*, 2007.
- [Greco et al.(2009)] G. Greco, E. Malizia, L. Palopoli, F. Scarcello. On the complexity of compact coalitional games. In *Proc. IJCAI'09*, 2009.
- [Mas-Colell et al.(1995)] Mas-Colell, A., Whinston, M., Green, J. Microeconomic Theory. Oxford University Press, 1995.
- [Matsui and Matsui(2000)] Y. Matsui and T. Matsui. A survey of algorithms for calculating power indices of weighted majority games. *Jnl of the Oper. Res. Soc. of Japan*, 43:71–86, 2000.
- [Moldovanu and Winter(1995)] Moldovanu, B., Winter, E. Order Independent Equilibria. *Games and Economic Behavior* 9:21–34, 1995.
- [Myerson(1991)] Myerson, R. Game Theory: Analysis of Conflict. Harvard University Press, 1991.
- [Myerson(1984)] Myerson, R. B. Cooperative games with incomplete information. *Int. J. Game Theory* 13:69–86, 1984.
- [Myerson(2007)] Myerson, R. B. Virtual utility and the core for games with incomplete information. *Journal of Economic Theory* 136(1):260–285, 2007.
- [Ohta et al.(2008)] Ohta, N., Conitzer, V., Satoh, Y., Iwasaki, A., Yokoo, M. Anonymity-proof Shapley value: extending Shapley value for coalitional games in open environments. In *Proc. AAMAS'08*, 2008.
- [Okada(1996)] Okada, A. A Noncooperative Coalitional Bargaining Game With Random Proposers. *Games and Economic Behavior* 16:97–108, 1996.
- [Osborne and Rubinstein(1994)] Osborne, M., Rubinstein, A. A course in game theory. MIT Press, 1994.
- [Perry and Reny(1994)] Perry, M., Reny, P. A Noncooperative View of Coalition Formation and the Core. *Econometrica* 62(4):795–817, 1994.
- [Rahwan and Jennings(2007)] Rahwan, T., Jennings, N. R. An algorithm for distributing coalitional value calculations among cooperating agents. *Artificial Intelligence* 171(8-9):535–567, 2007.

- [Rahwan and Jennings(2008)] Rahwan, T., Jennings, N. R. An improved dynamic programming algorithm for coalition structure generation. In *Proc. AAMAS'08*, 2008.
- [Rapoport(1970)] Rapoport, A. *N-Person Game Theory*. University of Michigan Press, 1970.
- [Rothkopf et al.(1995)] Rothkopf, M.H., Pekec, A., Harstad, R. M. Computationally manageable combinatorial auctions. *Management Science* 44(8):1131–1147, 1995.
- [Rubinstein(1982)] Rubinstein, A. Perfect Equilibrium in a Bargaining Model. *Econometrica* 50(1):97–110, 1982.
- [Sandholm et al.(1999)] Sandholm, T., Larson, K., Andersson, M., Shehory, O., Tohme, F. Coalition Structure Generation with Worst Case Guarantees. *Artificial Intelligence* 111(1–2):209–238, 1999.
- [Sandholm and Lesser(1997)] Sandholm, T., Lesser, V. Coalitions Among Computationally Bounded Agents. *Artificial Intelligence* 94(1):99–137, 1997.
- [Serrano and Vohra(1997)] Serrano, R., Vohra, R. Non-cooperative implementation of the core. *Social Choice and Welfare* 14:513–525, 1997.
- [Shapley(1953)] Shapley, L. A Value for n-Person Games. In: Kuhn, H., Tucker, A. (Eds.), *Contributions to the Theory of Games II*. Princeton University Press, Princeton, pp. 307–317, 1953.
- [Shapley(1967)] Shapley, L. On Balanced Sets and Cores. *Naval Research Logistics Quarterly* 14:453–460, 1967.
- [Shapley and Shubik(1972)] L. S. Shapley and M. Shubik. The assignment game. *Int Jnl of Game Theory*, 1:111–130, 1972.
- [Shehory and Kraus(1995)] Shehory, O., Kraus, S., 1995. Task allocation via coalition formation among autonomous agents. In *Proc. IJCAI'95*, 1995.
- [Shehory and Kraus(1998)] Shehory, O., Kraus, S. Methods for Task Allocation via Agent Coalition Formation. *Artificial Intelligence* 101(1–2):165–200, 1998.
- [Shehory and Kraus(1999)] Shehory, O., Kraus, S. Feasible Formation of Coalitions among Autonomous Agents in Nonsuperadditive Environments. *Computational Intelligence* 15:218–251, 1999.
- [Shehory et al.(1997)] Shehory, O., Sycara, K., Jha, S. Multiagent Coordination through Coalition Formation. *Agent Theories, Architectures and Languages*, 1997.
- [Solymosi and Raghavan(1994)] T. Solymosi and T. E. S. Raghavan, An algorithm for finding the nucleolus of assignment games, *Int Jnl of Game Theory*, 1:119–143, 1994.

- [Suijs and Borm(1999)] Suijs, J., Borm, P. Stochastic cooperative games: superadditivity, convexity and certainty equivalents. *Games & Econ. Beh.* 27:331–345, 1999.
- [Suijs et al.(1999)] Suijs, J., Borm, P., Wagenaere, A. D., Tijs, S. Cooperative games with stochastic payoffs. *European Journal of Operational Research* 113:193–205, 1999.
- [Taylor and Zwicker(1993)] A. Taylor and W. Zwicker. Weighted voting, multicameral representation, and power. *Games & Econ. Beh.*, 5:170–181, 1993.
- [Taylor and Zwicker(1999)] A. D. Taylor and W. S. Zwicker. *Simple Games*. Princeton University Press, 1999.
- [von Neumann and Morgenstern(1944)] von Neumann, J., Morgenstern, O. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, 1994.
- [Wooldridge and Dunne(2004)] M. Wooldridge and P. E. Dunne. On the computational complexity of qualitative coalitional games. *Artificial Intelligence* 158(1):27–73, 2004.
- [Wooldridge and Dunne(2006)] M. Wooldridge and P. E. Dunne. On the computational complexity of coalitional resource games. *Artificial Intelligence* 170(10):835–871, 2006.
- [Yan(2003)] Yan, H. Noncooperative selection of the core. *International Journal of Game Theory* 31(4):527–540, 2003.
- [Yeh(1986)] Yeh, D.Y. A dynamic programming approach to the complete set partitioning problem. *BIT Numerical Mathematics* 26(4):467–474, 1986.
- [Yeung et al. (1999)] C. Yeung and A. Poon and F. Wu Game-theoretical multi-agent modeling of coalition formation among autonomous agents *IEEE Transactions on Power Systems* 4(3), August 1999.
- [Yokoo et al.(2005)] Yokoo, M., Conitzer, V., Sandholm, T., Ohta, N., Iwasaki, A. Coalitional Games in Open Anonymous Environments. In *Proc. AAAI’05*, 2005.